About Fibonacci numbers and functions

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***Abstract*—** Fibonacci numbers and functions are topics of major interest in mathematics, due to the importance of their applications in many sciences. In the first part of this article we present some congruences involving Fibonacci and Lucas numbers. In the second one we discuss the dimensions of the Fibonacci numbers, defined on different closed intervals, starting with the evaluation of the Box dimension of this function defined on [0, 1] .

***Keywords*—**Fibonacci numbers, Fibonacci functions, quadratic residues, fractal dimension.

I. Introduction

The Fibonacci sequence was the outcome of a mathematical problem about rabbit breeding that was posed in the Liber Abaci (published in 1202) by Leonardo Fibonacci. [16]

The Fibonacci numbers are Nature's numbering system. They appear everywhere in Nature, from the leaf arrangement in plants, to the pattern of the florets of a flower, the bracts of a pinecone, or the scales of a pineapple.

Johann Kepler studied the Fibonacci sequence. Émile Léger appears to have been the first (or second, if the work of de Lagny is counted) to recognise that the worst case of the Euclidean algorithm occurs when the inputs are consecutive Fibonacci numbers. [10]

In 1843, Jacques Philippe Marie Binet discovered a formula for finding the nth term of the Fibonacci series. [17]

In 1870, the French mathematician Edouard Lucas gave different results on Fibonacci numbers and proved that that number $2^{127}-1$ is a prime one. The related Lucas sequences and Lucas numbers are named after him.

The golden ratio is closely connected to the Fibonacci series, being the limit of the sequence of the ratios of two successive Fibonacci numbers. There are no extant records of the Greek architects' plans for their most famous temples and buildings. So we do not know if they deliberately used the golden section in their architectural plans. The American mathematician Mark Barr used the Greek letter phi (φ) to represent the golden ratio, using the initial letter of the Greek Phidias who used the golden ratio in his sculptures. [17]

More extended information on the history of Fibonacci numbers, the golden ration and the relation between them can be found in [6] [14].

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Due to their importance in the applied sciences, the Fibonacci and Lucas numbers and their properties have been extensively studied in the last years. Generalizations, as Hyperfibonacci and Hyperlucas numbers have been done [2][7][8] and congruences modulo different numbers have been proved or conjectured. [11] [12][13][15]

The present article is organized as follows. In the next chapter we give some basic definitions and results, which will be used in the chapters III – V. In Chapter III we prove some congruences satisfied by the Fibonacci numbers F_p and Lucas numbers L_p, where p is a prime odd integer. The techniques employed are combinatorial or of elementary number theory.

In Chapter IV we determine an upper bound of the Box-dimension of the Fibonacci function defined on the closed interval [0, 1]. In the last chapter we present the results of the evaluation of different types of dimensions of the same function, using the software Benoit 1.3.1.

II. Mathematical Backgound

Let $(F_n)_{n\geq 0}$ be the sequence defined by:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,\ n \geq 0,$$

called *the Fibonacci sequence* and $(L_n)_{n\geq 0}$,

$$L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n,\ n \geq 0,$$

be *the Lucas sequence* and

$$f(x) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{2}{1+\sqrt{5}}\right)^x \cos(\pi x)\right\},\ x \in \mathrm{R}$$

be *the Fibonacci function* (Fig.1).

It is known that

$$F_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

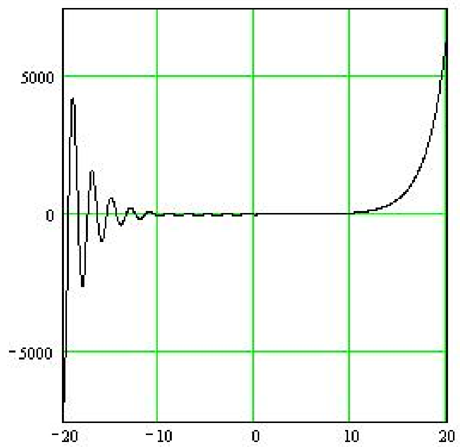

Fig.1. Graph of the Fibonacci function defined on [-20, 20]

Let p be a prime odd integer, n - an odd integer and a a natural number. The following notation will be used in the following:

- $\left(\frac{a}{p}\right)$ - *the Legendre symbol*;
- $\left(\frac{a}{n}\right)$ - *the Jacobi quadratic symbol*.

We recall that if p is an odd prime integer and a is an integer number, the Legendre symbol is defined by:

$$\left(\frac{a}{p}\right)=\begin{cases}+1, \text{ if } (a,p)=1 \text{ and} \\ \quad a \text{ is a quadratic residue mod } p \\ -1, \text{ if } (a,p)=1 \text{ and} \\ \quad a \text{ is not a quadratic residue mod } p \\ 0, \text{ if } p|a\end{cases}$$

and if n is an odd natural number, $n = p_1 \cdot p_2 \cdot ... \cdot p_r$, where $p_1, p_2, ..., p_r$ are prime integers, then the quadratic Jacobi symbol is defined by:

$$\left(\frac{a}{n}\right)=\left(\frac{a}{p_1}\right)\cdot\left(\frac{a}{p_2}\right)\cdot ... \cdot\left(\frac{a}{p_r}\right).$$

The following result about the Fibonacci and Lucas numbers is known:

Theorem 2.1 (Legendre, Lagrange). *Let p be a prime odd integer. Then the Fibonacci number, F_p, has the property:*

$$F_p \equiv \left(\frac{p}{5}\right) (mod\ p)$$

and the Lucas number, L_p, satisfies:

$$L_p \equiv 1\ (mod\ p).$$

We remember the following theorems that will be used to prove the results in the next chapter:

Fermat's little theorem. *If p is a prime number, then for any integer a, $a^p - a$ is evenly divisible by p.*

Remark. A variant of this theorem is the following: *If p is a prime and a is an integer coprime to p, then*

$$a^{p-1} \equiv 1\ (mod\ p).$$

Euler's criterion. *Let p be an odd prime and a an integer coprime to p. Then*

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}}\ (mod\ p).$$

The quadratic reciprocity law [9]. *Let p and q be two distinct odd prime integers. Then*

$$\left(\frac{p}{q}\right)\cdot\left(\frac{q}{p}\right)=(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

The fractal character [3] [4] of Fibonacci series and function may be studied by calculating different types of dimensions (ruler, box, information, Hausdorff) or the Hurst coefficient [1].

Let F be a nonempty and bounded subset of $\mathbf{R}^2$, $N_\delta(F)$ the least number of sets whose union covers F with and diameters that do not exceed a given $\delta > 0$. The *upper bound box dimension of F* is defined by:

$$\overline{\dim}_B F = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta},$$

and *the lower bound box dimension of F*, by:

$$\underline{\dim}_B F = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

If $\overline{\dim}_B F = \underline{\dim}_B F$, the common value is called the *box-dimension of F* and is denoted by $\dim_B F$ or $D_b(F)$.

Let $f: I \to \mathbf{R}$ be a function defined on the interval I and $[t_1, t_2]$ be a subinterval of I. We denote by:

$$R_f(t_1, t_2) = \sup_{t_1 \le t, u \le t_2} |f(t) - f(u)|.$$

and by $\Gamma(f)$ the graph of f.

Lemma 2.1 [5] *Let f be a continuous function defined on* $[0,1]$, $0 < \delta < 1$, *and m be the least integer greater than or equal to* $1/\delta$. *If* N_δ *is the number of the squares of the* $\delta -$ *mesh that intersect* $\Gamma(f)$. *Then:*

$$\delta^{-1} \sum_{j=0}^{m-1} R_f[j\delta, (j+1)\delta] \le N_\delta \le 2m + \delta^{-1} \sum_{j=0}^{m-1} R_f[j\delta, (j+1)\delta].$$

Remark. In practice, the box dimension is determined to be the exponent D_b such that $N_d \approx d - D_d$, where N_d is the number of boxes of linear size *d* necessary to cover a data set of points distributed in a two-dimensional plane.

To measure D_b one counts the number of boxes of linear size *d* necessary to cover the set for a range of values of *d* and plot the logarithm of N_d on the vertical axis versus the logarithm of *d* on the horizontal axis. If the set is indeed fractal, this plot will follow a straight line with a negative slope that equals D_b.

A choice to be made in this procedure is the range of values of *d*. In Benoit 1.3.1 software, a conservative choice may be to use as the smallest *d* ten times the smallest distance between points in the set, and as the largest *d* the maximum distance between points in the set divided by ten. We shall discuss this aspect, in detail in the last chapter.

The Hurst exponent can be calculated by rescaled range analysis (R/S analysis), by the following algorithm [1].

A time series $(X_k)_{k \in \overline{1,N}}$ is divided into *d* sub-series of length *m*. For each sub-series *n* = 1, … , *d* :

- Find the mean, E_n and the standard deviation, S_n;
- Normalize the data (X_{in}) by subtracting the sub-series mean:

$$Z_{in} = X_{in} - E_n,\ i = 1,...,m;$$

- Create a cumulative time series:

$$Y_{in} = \sum_{j=1}^{i} Z_{jn},\ i = 1,...,m;$$

- Find the range:

$$R_n = \max_{j=\overline{1,m}} Y_{jn} - \min_{j=\overline{1,m}} Y_{jn};$$

- Rescale the range R_n / S_n;
- Calculate the mean value of the rescaled range for all sub-series of length *m*:

$$(R/S)_m = \frac{1}{d} \sum_{n=1}^{d} R_n / S_n.$$

Hurst found that (*R/S*) scales by power - law as time increases, which indicates $(R/S)_t = c \cdot t^H$.

In practice, in classical *R/S* analysis, *H* can be estimated as the slope of log-log plot of $(R/S)_t$ versus *t*.

The fractal dimension of the trace can then be calculated from the relationship between the Hurst exponent H and *the fractal dimension:* $D_{rs} = 2 - H$, where D_{rs} denotes the fractal dimension estimated from the rescaled range method.

III. Congruences of Fibonacci numbers

Lemma 3.1 *Let n be a positive integer,* $n \ge 2$ *and p a prime odd integer. Then p divides* $\binom{p}{np} - n$.

Proof.

The proof is simple, using the congruence:

$$\binom{pb}{pa} \equiv \binom{b}{a}\ (mod\ p),$$

with *a* and *b* - positive integers, $b \le a$ and *p* - a prime integer, or using induction after $n \in \mathbf{N}^*$ and the identity

$$\sum_{k=0}^{m} \binom{n}{k} \binom{t}{m-k} = \binom{n+t}{m}.$$

Lemma 3.2 *Let p be a prime odd integer. Then p divides* $\binom{2p}{k}$, *for* $1 \le k \le 2p - 1$, $k \ne p$.

Proof.

$$\binom{2p}{k} = \frac{(2p)!}{k!(2p-k)!} = \frac{1 \cdot 2 \cdot ... \cdot p \cdot ... \cdot (2p)}{k!(2p-k)!}.$$

If $1 \le k \le p - 1$, it results that *p* does not divide *k*! and

$$p + 1 \le 2p - k \le 2p - 1.$$

This implies that:

$p|(2p - k)!$ and p^2 doesn't divide $(2p - k)!$

Thus: $p \left| \binom{2p}{k} \right.$.

If $p + 1 \le k \le 2p - 1$, it results that

$$p \mid k! \text{ and } p^2 \text{ doesn't divide } k!.$$

We have $1 \le 2p - k \le p - 1$, so

$$p \text{ doesn't divide } (2p - k)!.$$

Therefore, $p \left| \binom{2p}{k} \right.$.

Proposition 3.3 *Let p be a prime odd integer. Then the Lucas number* $L_{2p} \equiv 3 \pmod p$.

Proof.

$$L_{2p} = \left(\frac{1+\sqrt{5}}{2}\right)^{2p} + \left(\frac{1-\sqrt{5}}{2}\right)^{2p} \Leftrightarrow$$

$$L_{2p} = \frac{1}{2^{2p-1}} \cdot \sum_{\substack{k=0 \\ k\ even}}^{2p} \binom{2p}{k} \cdot 5^{\frac{k}{2}}.$$

Using Lemma 3.2, we obtain that

$$L_{2p} \equiv \frac{1}{2^{2p-1}} \cdot \left(1+5^p\right) \pmod p \Leftrightarrow$$

$$L_{2p} \equiv \frac{2}{4^p} \cdot \left(1+5^p\right) \pmod p.$$

Applying Fermat's small theorem, we obtain:

$$L_{2p} \equiv 3 \pmod p.$$

Proposition 3.4 *Let p be a prime odd positive integer. Then, the Fibonacci number*

$$F_{2p} \equiv \left(\frac{p}{5}\right) \pmod p.$$

Proof.

$$F_{2p} = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^{2p} - \left(\frac{1-\sqrt{5}}{2}\right)^{2p}\right] \Leftrightarrow$$

$$F_{2p} = \frac{1}{2^{2p-1}} \cdot \sum_{\substack{k=0 \\ k\ odd}}^{2p} \binom{2p}{k} \cdot 5^{\frac{k-1}{2}}.$$

Using Lemma 3.2 and Lemma 3.1 we obtain that

$$F_{2p} \equiv \frac{1}{4^{p-1}} \cdot 5^{\frac{p-1}{2}} \pmod p.$$

Applying Fermat's little theorem and Euler's criterion, it results that:

$$F_{2p} \equiv \left(\frac{5}{p}\right) \pmod p.$$

Applying the quadratic reciprocity law, we have:

$$\left(\frac{p}{5}\right) \cdot \left(\frac{5}{p}\right) = (-1)^{p-1} = 1 \Leftrightarrow \left(\frac{5}{p}\right) = \left(\frac{p}{5}\right),$$

so

$$F_{2p} \equiv \left(\frac{p}{5}\right) \pmod p.$$

Corollary 3.5 *Let p be a prime odd integer. Then*:

$$p \left| \left(F_{2p-\left(\frac{p}{2}\right)} - 1\right) \right..$$

Proof.

Since between Fibonacci and Lucas numbers there is the relationship

$$L_n = F_{n+1} + F_{n-1},$$

it results that

$$L_{2p} = F_{2p+1} + F_{2p-1}.$$

Applying Proposition 3.3, Proposition 3.4 and the properties of Fibonacci and Lucas numbers, we obtain:

$$F_{2p-1} = \frac{L_{2p} - F_{2p}}{2} \equiv \frac{3 - \left(\frac{p}{5}\right)}{2} \pmod p,$$

$$F_{2p+1} = \frac{L_{2p} + F_{2p}}{2} \equiv \frac{3 + \left(\frac{p}{5}\right)}{2} \pmod p.$$

Using the properties of Legendre' symbol it results that:

$$p \left| \left(F_{2p-\left(\frac{p}{5}\right)} - 1\right) \right..$$

IV. On the Box Dimension of Fibonacci Functions Defined on $[0, 1]$

In the following we shall use Lemma 2.1, to estimate the upper bound of the Fibonacci function *f*, defined on [0, 1].

Theorem 4.1. *The upper Box dimension of the graph of the Fibonacci function defined on* [0, 1] *is less or equal than* 1.

Proof.

In order to simplify the calculation we denote by $a = \frac{1+\sqrt{5}}{2}$.

If $0 < \delta < 1$, then:

$$m - 1 < \frac{1}{\delta} < m = \left[\frac{1}{\delta}\right] + 1 \le \frac{1}{\delta} + 1 .$$

We evaluate the difference $\left| f\left((j+1)\delta\right) - f\left(j\delta\right)\right|$, in order to apply Lemma 1.

$$\left| f\left((j+1)\delta\right) - f\left(j\delta\right)\right| =$$

$$= \frac{1}{\sqrt{5}}\left| a^{(j+1)\delta} - \frac{1}{a^{(j+1)\delta}}\cos\left(\pi\left(j+1\right)\delta\right) - a^{j\delta} + \frac{1}{a^{j\delta}}\cos\left(\pi j\delta\right)\right| =$$

$$= \frac{1}{\sqrt{5}}\left| a^{j\delta}\left(a^{\delta} - 1\right) - \frac{1}{a^{j\delta}}\left(\frac{\cos\left(\pi\left(j+1\right)\delta\right)}{a^{\delta}} - \cos\left(\pi j\delta\right)\right)\right| .$$

Since

$$0 < \delta < 1,\ a > 1 \Rightarrow 1 < a^{\delta} < a \Rightarrow \begin{cases} a^{\delta} - 1 > 0 \\ \frac{1}{a^{\delta}} < 1 \end{cases}$$

and

$$\left|\cos x - \cos y\right| \le \left|x - y\right|, (\forall)x, y \in \mathbf{R},$$

then

$$\left| f\left((j+1)\delta\right) - f\left(j\delta\right)\right| \le$$

$$\le \frac{1}{\sqrt{5}}\left[a^{j\delta}\left(a^{\delta} - 1\right) + \frac{1}{a^{j\delta}}\left|\cos\left(\pi\left(j+1\right)\delta\right) - \cos\left(\pi j\delta\right)\right|\right] \le$$

$$\le \frac{1}{\sqrt{5}}\left[a^{j\delta}\left(a^{\delta} - 1\right) + \frac{1}{a^{j\delta}}\left|\pi\left(j+1\right)\delta - \pi j\delta\right|\right] \Leftrightarrow$$

$$\left| f\left((j+1)\delta - f\left(j\delta\right)\right)\right| \le \frac{1}{\sqrt{5}}\left[a^{j\delta}\left(a^{\delta} - 1\right) + \frac{1}{a^{j\delta}}\pi\delta\right] .$$

Applying Lemma 2.1 we obtain:

$$N_{\delta} \le 2m + \delta^{-1}\sum_{j=0}^{m-1} R_f\left[j\delta, (j+1)\delta\right] \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\sum_{j=0}^{m-1}\left[a^{j\delta}\left(a^{\delta} - 1\right) + \frac{\pi\delta}{a^{j\delta}}\right] \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left[\left(a^{\delta} - 1\right)\sum_{j=0}^{m-1} a^{j\delta} + \pi\delta\sum_{j=0}^{m-1}\frac{1}{a^{j\delta}}\right] \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{\delta} - 1\right)\left(1 + a^{\delta} + \ldots + a^{(m-1)\delta}\right) +$$

$$+ \frac{\pi}{\sqrt{5}}\left(1 + a^{-\delta} + \ldots + a^{-(m-1)\delta}\right) \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right) + \frac{\pi}{\sqrt{5}}\frac{1 - a^{-m\delta}}{1 - a^{-\delta}} \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right) + \frac{\pi}{\sqrt{5}}\frac{1 - \frac{1}{a^{m\delta}}}{1 - \frac{1}{a^{\delta}}} \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right) + \frac{\pi}{\sqrt{5}}\frac{1 - \frac{1}{a^{m\delta}}}{1 - \frac{1}{a^{\delta}}} \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right) + \frac{\pi}{\sqrt{5}}\frac{\frac{a^{m\delta} - 1}{a^{m\delta}}}{\frac{a^{\delta} - 1}{a^{\delta}}} \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right) + \frac{a^{m\delta} - 1}{(a^{\delta} - 1)a^{(m-1)\delta}}\frac{\pi}{\sqrt{5}} \Leftrightarrow$$

$$N_{\delta} \le 2m + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right)\left[1 + \frac{\pi\delta}{(a^{\delta} - 1)a^{(m-1)\delta}}\right] \le$$

$$\le 2\left(\frac{1}{\delta} + 1\right) + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right)\left[1 + \frac{\pi\delta}{(a^{\delta} - 1)a^{(m-1)\delta}}\right] =$$

$$= 2\frac{\delta + 1}{\delta} + \frac{1}{\delta\sqrt{5}}\left(a^{m\delta} - 1\right)\left[1 + \frac{\pi\delta}{(a^{\delta} - 1)a^{(m-1)\delta}}\right] =$$

$$=\frac{1}{\delta}\left\{2(\delta+1)+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\right\}.$$

$$\log N_{\delta}\leq$$

$$\leq\log\left\{2\delta+2+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\right\}-\log\delta.$$

$$\frac{\log N_{\delta}}{-\log\delta}\leq$$

$$\leq\frac{\log\left\{2\delta+2+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\right\}-\log\delta}{-\log\delta}\Rightarrow$$

$$\frac{\log N_{\delta}}{-\log\delta}\leq 1-\frac{\log\left\{2\delta+2+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\right\}}{\log\delta}.$$

$$2\delta+2+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\leq$$

$$\leq 2\delta+2+\frac{1}{\sqrt{5}}\left(a^{\left(\frac{1}{\delta}+1\right)\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]$$

$$\leq 2\delta+2+\frac{1}{\sqrt{5}}\left(a^{\delta+1}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{1-\delta}}\right],$$

because:

$$\frac{1}{\delta}-1<m-1\leq\frac{1}{\delta}\Leftrightarrow\frac{1-\delta}{\delta}<m-1\leq\frac{1}{\delta}\Rightarrow$$

$$1-\delta<(m-1)\delta\leq 1\Rightarrow$$

$$a^{1-\delta}<a^{(m-1)\delta}\leq a\Rightarrow\frac{1}{a}\leq\frac{1}{a^{(m-1)\delta}}\leq\frac{1}{a^{1-\delta}}.$$

But,

$$\lim_{\delta\to 0}\frac{\delta}{a^{\delta}-1}=\frac{1}{\ln a},\ \lim_{\delta\to 0}a^{1-\delta}=a,$$

So,

$$\lim_{\delta\to 0}\left\{2\delta+2+\frac{1}{\sqrt{5}}\left(a^{m\delta}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{(m-1)\delta}}\right]\right\}\leq$$

$$\leq\lim_{\delta\to 0}\left\{2\delta+2+\frac{1}{\sqrt{5}}\left(a^{\delta+1}-1\right)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{1-\delta}}\right]\right\}\leq$$

$$\leq 2+\frac{1}{\sqrt{5}}(a-1)\left(1+\frac{\pi}{a\ln a}\right)\Rightarrow$$

$$\lim_{\delta\to 0}\frac{\log(N_{\delta})}{-\log\delta}\leq$$

$$\leq 1-\lim_{\delta\to 0}\frac{\log\left(2\delta+2+\frac{1}{\sqrt{5}}(a^{\delta+1}-1)\left[1+\frac{\pi\delta}{(a^{\delta}-1)a^{1-\delta}}\right]\right)}{\log\delta}\leq$$

$$\leq 1-\log\left\{2+\frac{1}{\sqrt{5}}(a-1)\left(1+\frac{\pi}{a\ln a}\right)\right\}\lim_{\delta\to 0}\frac{1}{\log\delta}=1.$$

Thus,

$$\lim_{\delta\to 0}\frac{\log(N_{\delta})}{-\log\delta}\leq 1$$

and $\overline{\dim}_B\Gamma(f)\leq 1$.

V. Estimations of Dimensions of Fibonacci Function Defined on a Symmetric Interval

It is well known that the Fibonacci sequence have fractal properties.

The software used was Benoit 1.3.1.

In the following we determine the box - dimension and the fractal dimension D_{rs} of the Fibonacci function, defined on a domain $[-a, a]$, where a > 0. For exemplification, a has be chosen to be 20.

In theory, to determine the box – dimension, for each box size, the grid should be overlaid in such a way that the minimum number of boxes is occupied. This is accomplished in Benoit by rotating the grid for each box size through 90 degrees and plotting the minimum value of N_d.

Benoit permits the user to select the angular increments of rotation. For our calculation, the increment of the rotation angle has been set to 15 degrees, the coefficient of box-decrease, 1.1, and the size of the largest box 70 pixels (Fig.3).

The value calculated for the box dimension was 1.91362. Looking to the left hand side of Fig. 3, we see that the chart of the number of boxes versus the boxes - side length is linear, proving that fractal character of the Fibonacci number defined on [-20, 20] .

For the same function the value of Hurst coefficient has been determined to be H = 1.536 and D_{rs} = 0.464 (Fig.4). Therefore, the function has the long range dependence property on the studied interval.

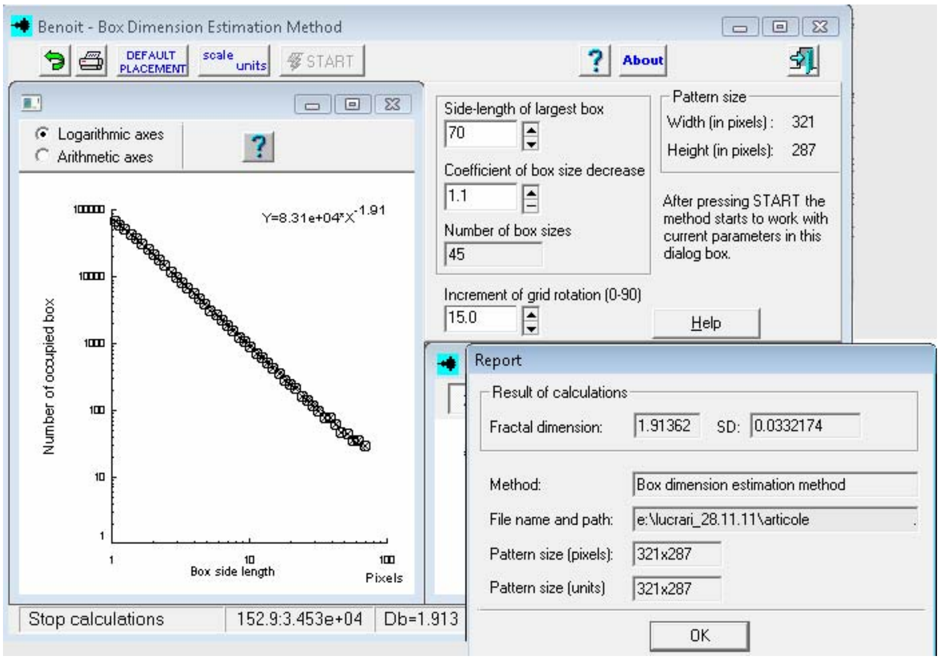

Fig.3. Determination of the box-dimension of the Fibonacci function defined on [-20, 20]

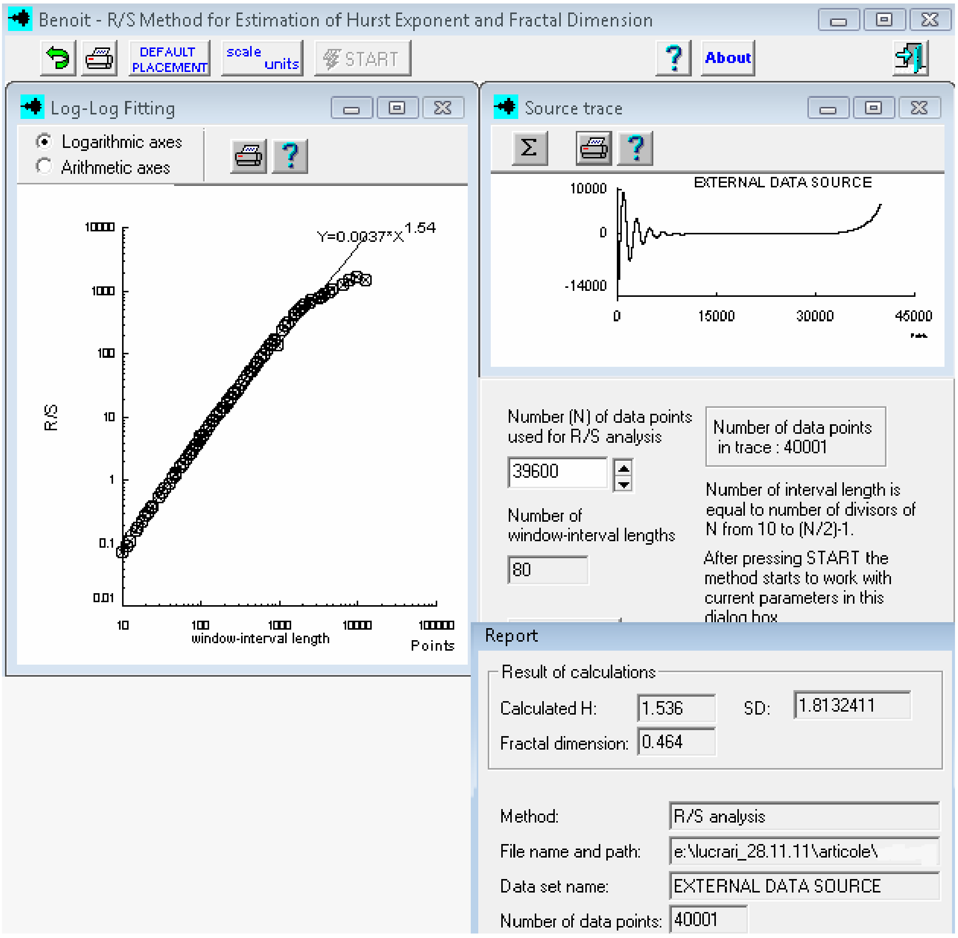

Fig.4. Determination of the Hurst coefficient for the Fibonacci function defined on [-20, 20]

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