Generating passive systems from recursively defined polynomials

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Abstract— In single-input single-output linear time-invariant systems, and in particular in filter modelling and design, some properties of the system (such as shaped frequency responses) are achieved by using polynomials of particular types for the design of the system transfer function. In this paper it is shown that single-input single-output linear time-invariant systems having as transfer function the ratio between two successive polynomials recursively defined have particular properties. In particular, if the ratio between successive polynomials of either Fibonacci or Lucas type is considered, loss-less systems are achieved, while starting from Jacobsthal or Morgan-Voyce polynomials relaxation systems are defined. Furthermore, it is also shown that transfer functions obtained by considering either the ratio between a Fibonacci polynomial of order n and a Lucas polynomial of order n or between a Fibonacci polynomial of order n and a Lucas polynomial of order $n-1$ multiplied by the complex variable s are also loss-less.

I. Introduction

Some of the properties of single-input single-output time-invariant linear systems are connected to the characteristics of the polynomials involved in their transfer function. This is particularly important for filter modelling and design. If the polynomials at the numerator and/or denominator of the transfer function of a single-input single-output (SISO) linear time-invariant (LTI) system belong to special classes of polynomials, then the system may have particular features. For instance, in filter design [1], Butterworth polynomials are used to implement filters with maximally flat approximation at dc, Chebyshev polynomials of the first kind [2] to minimize the maximum deviation from the ideal flat characteristic in the bandpass and Bessel polynomials to implement filters with a maximally linear phase response. Another example are Laguerre filters constituting an orthonormal basis for the Hilbert space, and for this used in system identification and reduced-order modelling [3]. Additionally, it has been demonstrated that systems having as transfer function the sum of the first $n+1$ Laguerre functions have all the singular values equals each other [4]. In general, filter design is very important both in the analog and in the digital word [1], [5], [6], [7].

In this paper we study recursively defined polynomials such as Fibonacci, Lucas or Jacobsthal polynomials and show how from such polynomials systems with particular peculiarities can be defined. In particular, we show that the ratio between polynomials of Fibonacci or Lucas type leads to the definition of a loss-less system (in particular, this hold for two successive either Fibonacci or Lucas polynomials, for the ratio between a Fibonacci polynomial of order n and a Lucas polynomial of order n and for the ratio between a Fibonacci polynomial of order n and a Lucas polynomial of order $n-1$ multiplied by the complex variable s), while the ratio between successive Jacobsthal or Morgan-Voyce polynomials leads to positive real relaxation systems.

Loss-less systems and positive real relaxation systems (or shortly relaxation systems) [9], [10], [11] are particular classes of passive systems. Loss-less systems are characterized by the fact that, while in a passive one-port circuit the total energy delivered to it from any generating source connected to it, is always non-negative, in a loss-less one-port circuit, when a finite amount of energy is put on its elements, all the energy can be extracted again. Given a linear time-invariant system defined by:

$$\begin{array}{l}\dot{\mathbf{x}}(t) = \mathrm{A}\mathbf{x}(t) + \mathrm{B}\mathbf{u}(t)\\ \mathbf{y}(t) = \mathrm{C}\mathbf{x}(t) + \mathrm{D}\mathbf{u}(t)\end{array} \quad (1)$$

where $\mathrm{A} \in \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{n\times m}$, $\mathrm{C} \in \mathbb{R}^{m\times n}$ and $\mathrm{D} \in \mathbb{R}^{m\times m}$, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$ the input and output vectors, and the transfer matrix is given by:

$$\mathrm{G}(s) = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D} \quad (2)$$

the necessary and sufficient conditions for the system to be loss-less are:

- all poles of $\mathrm{G}(s)$ are simple and have zero real part and the residue matrix at any pole is a non-negative definite matrix;
- $\mathrm{G}^T(-s) + \mathrm{G}(s) = 0$ for all s such that s is not a pole of any element of $\mathrm{G}(s)$.

In the case of SISO systems, the loss-lessness of the transfer function can be easily tested from the inspection of its zeros and poles. In fact, it is worth recalling that, as a consequence of Foster's reactance theorem [8], the poles and zeros of any loss-less function must alternate with increasing frequency. This criterion will be helpful in the following to prove the loss-lessness of the transfer function defined by Fibonacci/Lucas polynomials.

As far as concerns relaxation systems, these are passive systems characterized by the fact that they can be realized using only a subsets of the possible passive components (resistors, capacitors and inductors). In fact, while in general a passive system can be realized with an electrical circuit with resistors, capacitors and inductors, a relaxation system

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can be realized by only using either resistors and capacitors or resistors and inductors. Restricting the discussion to SISO systems, a simple necessary and sufficient condition is given by the following result [12]: a transfer function represents the impedance (the admittance) of a relaxation system (that can be implemented by only using resistors and capacitors) if and only if the poles and the zeros are all negative real, simple and alternate in frequency and the critical point closest to the origin is a pole (a zero). Another important result to be recalled is related to the continued fraction expansion of a relaxation system. We restrict the discussion to the most interesting cases for the results presented below. It can be demonstrated that a relaxation system $G(s)$ of order n with a pole at the origin can be rewritten as follows:

$$G(s) = \frac{h_0}{s} + \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cdots\cdots\cdots\cdots\cdots + \cfrac{1}{h_{2n+1}}}} \tag{3}$$

where h_i with $i = 0, \ldots, 2n{+}1$ are real positive coefficients. A relaxation system $G(s)$ of order n with a zero at the origin has the following continued fraction expansion:

$$G(s) = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cdots\cdots\cdots\cdots\cdots + \cfrac{1}{\frac{h_{2n+2}}{s}}}} \tag{4}$$

where h_i with $i = 1, \ldots, 2n{+}2$ are real positive coefficients. Finally, if the relaxation system has finite values at $s = 0$ and $s \to \infty$, then it can be rewritten as:

$$G(s) = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cdots\cdots\cdots\cdots\cdots + \cfrac{1}{h_{2n+1}}}} \tag{5}$$

where h_i with $i = 1, \ldots, 2n{+}1$ are real positive coefficients.

The remaining of the paper is organized as follows: in Section II the recursively defined polynomials that in the following will be used to generate loss-less or relaxation systems are briefly discussed along with some important properties; in Section III loss-less systems generated by recursively defined polynomials are discussed, while in Section IV the case of relaxation systems is dealt with. Section V concludes the paper.

II. Recursively defined polynomials

In this Section, we briefly recall the definition of the polynomials that will be used in the following, along with some of their most important properties. The polynomials are recursively defined as described in the following. In particular, we discuss Finonacci polynomials, Lucas polynomials, Jacobsthal polynomials and Morgan-Voyce polynomials [13].

A. Fibonacci polynomials

Fibonacci polynomials are a sequence of polynomials, recursively defined in a way analogous to the way in which Fibonacci numbers are defined [13]. Fibonacci polynomials in fact are recursively defined from $f_1(x) = 1$, $f_2(x) = x$ and from the following rule:

$$f_n(x) = x f_{n-1}(x) + f_{n-2}(x) \tag{6}$$

The first ten Fibonacci polynomials are for instance reported in Table I.

TABLE I
List of the first ten Fibonacci polynomials

Order	Fibonacci Polynomial
$n = 1$	$f_1(x) = 1$
$n = 2$	$f_2(x) = x$
$n = 3$	$f_3(x) = x^2 + 1$
$n = 4$	$f_4(x) = x^3 + 2x$
$n = 5$	$f_5(x) = x^4 + 3x^2 + 1$
$n = 6$	$f_6(x) = x^5 + 4x^3 + 3x$
$n = 7$	$f_7(x) = x^6 + 5x^4 + 6x^2 + 1$
$n = 8$	$f_8(x) = x^7 + 6x^5 + 10x^3 + 4x$
$n = 9$	$f_9(x) = x^8 + 7x^6 + 15x^4 + 10x^2 + 1$
$n = 10$	$f_{10}(x) = x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$

The degree of $f_n(x)$ is $n-1$. Odd Fibonacci polynomials contain only even powers of x, while even Fibonacci polynomials contain only odd powers of x.

Fibonacci polynomials have some interesting properties [13], [14], [15].

Property 1: The Fibonacci polynomials, evaluated at $x = 1$, give the Fibonacci numbers, i.e.,

$$f_n(1) = F_n \tag{7}$$

where F_n are the Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$

Property 2: $f(x)$ is given by the following explicit formula:

$$f_n(x) = \sum_{j=0}^{[(n-1)/2]} \left(\begin{array}{c} n-j-1 \\ j \end{array} \right) x^{n-2j-1} \tag{8}$$

where $[\cdot]$ is the floor function and $\left(\begin{array}{c} n-j-1 \\ j \end{array} \right)$ the binomial coefficient.

Property 3: $f_n(x)$ divides $f_m(x)$ if and only if n divides m.

Property 4: The roots of $f_n(x)$ are given by:

$$r_k = 2i \cos \frac{k\pi}{n} \tag{9}$$

for $k = 1, \ldots, n-1$.

B. Lucas polynomials

Lucas polynomials are also recursively defined polynomials [13], [16]. The recursive rule of Lucas polynomials is:

$$l_n(x) = xl_{n-1}(x) + l_{n-2}(x) \tag{10}$$

given $l_0(x) = 2$ and $l_1(x) = x$.

Table II reports the first ten Lucas polynomials.

TABLE II
LIST OF THE FIRST TEN LUCAS POLYNOMIALS

Order	Lucas Polynomial
$n = 1$	$l_1(x) = x$
$n = 2$	$l_2(x) = x^2 + 2$
$n = 3$	$l_3(x) = x^3 + 3x$
$n = 4$	$l_4(x) = x^4 + 4x^2 + 2$
$n = 5$	$l_5(x) = x^5 + 5x^3 + 5x$
$n = 6$	$l_6(x) = x^6 + 6x^4 + 9x^2 + 2$
$n = 7$	$l_7(x) = x^7 + 7x^5 + 14x^3 + 7x$
$n = 8$	$l_8(x) = x^8 + 8x^6 + 20x^4 + 16x^2 + 2$
$n = 9$	$l_9(x) = x^9 + 9x^7 + 27x^5 + 30x^3 + 9x$
$n = 10$	$l_{10}(x) = x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2$

Similarly to Fibonacci polynomials, the roots of Lucas polynomials can also be computed in closed form.

Property 5: The zeros of $l_n(x)$ are given by:

$$r_k = 2i \cos \frac{(2k+1)\pi}{2n} \tag{11}$$

for $k = 0, \ldots, n-1$.

C. Jacobsthal polynomials

A third class of recursively defined polynomials is represented by Jacobsthal polynomials. Such polynomials are defined by:

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x) \tag{12}$$

provided that $J_1(x) = 1$ and $J_2(x) = 1$.

Table III reports the first ten Jacobsthal polynomials.

TABLE III
LIST OF THE FIRST TEN JACOBSTHAL POLYNOMIALS

Order	Jacobsthal Polynomial
$n = 1$	$J_1(x) = 1$
$n = 2$	$J_2(x) = 1$
$n = 3$	$J_3(x) = x + 1$
$n = 4$	$J_4(x) = 2x + 1$
$n = 5$	$J_5(x) = x^2 + 3x + 1$
$n = 6$	$J_6(x) = 3x^2 + 4x + 1$
$n = 7$	$J_7(x) = x^3 + 6x^2 + 5x + 1$
$n = 8$	$J_8(x) = 4x^3 + 10x^2 + 6x + 1$
$n = 9$	$J_9(x) = x^4 + 10x^3 + 15x^2 + 7x + 1$
$n = 10$	$J_{10}(x) = 5x^4 + 20x^3 + 21x^2 + 8x + 1$

It is interesting to note that $J_{2n-1}(x)$ and $J_{2n}(x)$ have the same degree: the degree of $J_n(x)$ is in fact $[(n-1)/2]$. Furthermore, $J_n(x)$ and $f_n(x)$ have the same coefficients, but in reverse order.

Another class of polynomials, related to Jacobsthal polynomials, is defined by:

$$K_n(x) = K_{n-1}(x) + xK_{n-2}(x) \tag{13}$$

given $K_1(x) = 1$ and $K_2(x) = x$.

Table IV reports the first ten polynomials $K_n(x)$.

TABLE IV
LIST OF THE FIRST TEN $K_n(x)$ POLYNOMIALS

Order	$K_n(x)$
$n = 1$	$K_1(x) = 1$
$n = 2$	$K_2(x) = x$
$n = 3$	$K_3(x) = 2x$
$n = 4$	$K_4(x) = x^2 + 2x$
$n = 5$	$K_5(x) = 3x^2 + 2x$
$n = 6$	$K_6(x) = x^3 + 5x^2 + 2x$
$n = 7$	$K_7(x) = 4x^3 + 7x^2 + 2x$
$n = 8$	$K_8(x) = x^4 + 9x^3 + 9x^2 + 2x$
$n = 9$	$K_9(x) = 5x^4 + 16x^3 + 11x^2 + 2x$
$n = 10$	$K_{10}(x) = x^5 + 14x^4 + 25x^3 + 13x^2 + 2x$

Such polynomials have similar properties to those of Jacobsthal polynomials.

D. Morgan-Voyce polynomials

There are two types of Morgan-Voyce polynomials [13], usually indicated as $b_n(x)$ or $B_n(x)$. The polynomials $b_n(x)$ are recursively defined by the following relationships:

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \tag{14}$$

provided that $b_0(x) = 1$ and $b_1(x) = x + 1$, or by

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \tag{15}$$

with $B_0(x) = 1$ and $B_1(x) = x + 2$.

Tables V and VI report the first ten Morgan-Voyce polynomials $b(x)$ and $B(x)$.

Morgan-Voyce polynomials have a remarkable property for what follows. Their zeros can be explicitly calculated as formally stated in the following properties.

Property 6: The zeros of $b_n(x)$ are given by:

$$r_k = -4 \sin^2 \frac{(2k-1)\pi}{(4n+2)} \tag{16}$$

for $k = 0, \ldots, n-1$.

Property 7: The zeros of $B_n(x)$ are given by:

$$r_k = -4 \sin^2 \frac{k\pi}{(2n+2)} \tag{17}$$

for $k = 0, \ldots, n-1$.

III. Generating loss-less systems from Fibonacci and Lucas Polynomials

Theorem 1: $G(s) = \frac{f_n(s)}{f_{n+1}(s)}$ represents the transfer function of a controllable and observable loss-less system of order n.

Proof. The fact that the system with transfer function $G(s) = \frac{f_n(s)}{f_{n+1}(s)}$ is controllable and observable is a direct consequence of the divisibility property (property 3) of Fibonacci polynomials. Let's focus on the loss-less property.

Recall that a SISO LTI system is loss-less if and only if its zeros and poles are all on the imaginary axis and alternate in frequency. Therefore, to demonstrate it suffices to demonstrate that the zeros of $f_n(s)$ and $f_{n+1}(s)$ have zero real part and alternate.

Since the roots of $f_n(s)$ and $f_{n+1}(s)$ represent the zeros Z_h and poles P_h of $G(s)$, thanks to property 4, they can be explicitly calculated:

$$\begin{array}{ll} Z_h = 2i\cos\frac{h\pi}{n}, & 1 \le h \le n-1 \\ P_h = 2i\cos\frac{h\pi}{n+1}, & 1 \le h \le n \end{array} \tag{18}$$

Therefore, all the zeros and poles of $G(s)$ have zero real part.

Let us now rewrite Z_h and P_h as

$$\begin{array}{ll} Z_h = 2iz_h, & 1 \le h \le n-1 \\ P_h = 2ip_h, & 1 \le h \le n \end{array} \tag{19}$$

i.e., $z_h = \cos\frac{h\pi}{n}$ and $\cos\frac{h\pi}{n+1}$.

Since for $1 \le h \le n-1$

$$0 < \frac{h\pi}{n} < \pi$$

and

$$\frac{h\pi}{n+1} < \frac{h\pi}{n},$$

then

$$\cos\frac{h\pi}{n+1} > \cos\frac{h\pi}{n}$$

for $1 \le h \le n-1$ and thus

$$p_h > z_h \tag{20}$$

Analogously, since $\frac{n-1}{n} < \frac{n}{n+1}$, then

$$p_n < z_{n-1} \tag{21}$$

From Eqs. (20) and (21) it follows that

$$p_1 > z_1 > p_2 > z_2 > \ldots > p_{n-1} > z_{n-1} > p_n$$

which demonstrates that the zeros and poles of $G(s)$ alternate in frequency.

Furthermore, the fact that loss-less systems have an odd positive real function, ie. $G(s) = -G^T(-s)$ can be also shown taking into account that:

$$f_n(s)f_{n+1}(-s) = -f_n(-s)f_{n+1}(s)$$

since either $f_n(s)$ is odd and $f_{n+1}(s)$ is even, or $f_n(s)$ is even and $f_{n+1}(s)$ is odd.

So, one has:

$$\frac{f_n(s)}{f_{n+1}(s)} = -\frac{f_n(-s)}{f_{n+1}(-s)}$$

and thus $G(s) = -G^T(-s)$. ⋄

Example 1: As an example of Theorem 1, consider $G(s) = \frac{f_6(s)}{f_7(s)}$, i.e.

$$G(s) = \frac{s^5 + 4s^3 + 3s}{s^6 + 5s^4 + 6s^2 + 1}.$$

$G(s)$ can be factorized as follows:

$$\frac{s(s^2+1)(s^2+3)}{(s^2+0.1981)(s^2+1.555)(s^2+3.247)}$$

Clearly, this represents the transfer function of a loss-less system. For instance, the pole-zero map shows how the poles and zeros of this system are all on the imaginary axis and alternate. ⋄

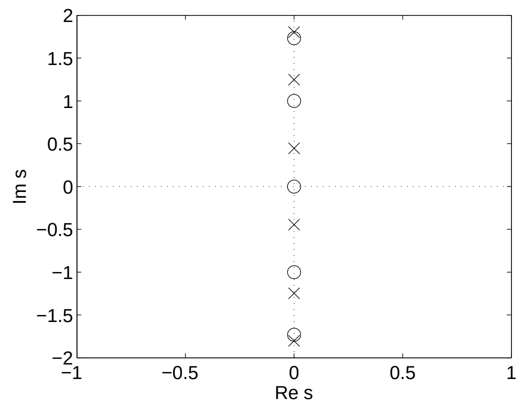

Fig. 1. Pole-zero map of system G(s) in Example 1.

For Lucas polynomials, a result analogous to Theorem 1 can be derived.

Theorem 2: $G(s) = \frac{l_{n-1}(s)}{l_n(s)}$ represents the transfer function of a controllable and observable loss-less system of order n.

Proof. The demonstration is analogous to that of Theorem 1. In fact, thanks to property 5, the zeros Q_h and poles T_h of $G(s)$ can be explicitly calculated:

$$\begin{array}{ll} Q_h = 2i\cos\frac{(2h+1)\pi}{2n} \triangleq 2iq_h, & 0 \le h \le n-1 \\ T_h = 2i\cos\frac{(2h+1)\pi}{2(n+1)} \triangleq 2it_h, & 0 \le h \le n \end{array} \tag{22}$$

All the poles and zeros of $G(s)$ thus have zero real part. Moreover, they alternate in frequency. In fact, since for $1 \le h \le n-1$

$$0 < \frac{(2h+1)\pi}{2n} < \pi$$

and

$$\frac{(2h+1)\pi}{2(n+1)} < \frac{(2h+1)\pi}{2n}$$

then

$$\cos\frac{(2h+1)\pi}{2(n+1)} > \cos\frac{(2h+1)\pi}{n}.$$

Moreover, since $\frac{2n-1}{2n} < \frac{2n+1}{2n+2}$, then $q_{n-1} < t_n$, and in conclusion:

$$t_1 > q_1 > t_2 > q_2 > \ldots > t_{n-1} > q_{n-1} > t_n$$

which demonstrates that the poles T_h and zeros Q_h of $G(s) = \frac{l_{n-1}(s)}{l_n(s)}$ alternate in frequency. ⋄

Example 2: As an example of Theorem 2, consider $G(s) = \frac{l_7(s)}{l_8(s)}$, i.e.

$$G(s) = \frac{s^7 + 7s^5 + 14s^3 + 7s}{s^8 + 8s^6 + 20s^4 + 16s^2 + 2}.$$

$G(s)$ can be factorized as follows:

$$\frac{s(s^2+0.753)(s^2+2.445)(s^2+3.802)}{(s^2+0.1522)(s^2+1.235)(s^2+2.765)(s^2+3.848)}$$

This is the transfer function of a loss-less system. For instance, the pole-zero map shown in Fig. 2 illustrates how the poles and zeros of this system are all on the imaginary axis and alternate. ⋄

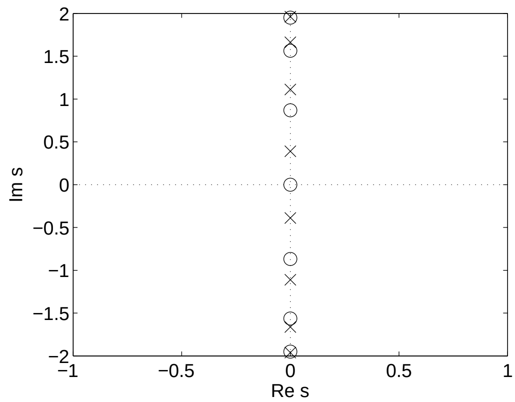

Fig. 2. Pole-zero map of system $G(s)$ in Example 2.

It is interesting to note that it is not possible to generate transfer functions of loss-less systems if the Fibonacci/Lucas polynomials are not successive. For instance, if we consider $G(s) = \frac{f_n(s)}{f_{n+2}(s)}$, since this does not have relative degree equal to one, it cannot represent the transfer function of a loss-less system. However, other loss-less systems can be generated by considering the ratio between Lucas and Fibonacci polynomials.

Keeping in mind the condition on the relative degree, one can consider either $G(s) = \frac{l_n(s)}{f_{n+2}(s)}$ or $G(s) = \frac{f_n(s)}{l_n(s)}$. The next example shows that $G(s) = \frac{l_n(s)}{f_{n+2}(s)}$ does not lead to loss-less systems, while Theorem 3 shows that $G(s) = \frac{f_n(s)}{l_n(s)}$ does.

Example 3: Consider $G(s) = \frac{l_4(s)}{f_6(s)} = \frac{s^4+4s^2+2}{s^5+4s^3+3s}$. The system is not loss-less, since $G(s) = \frac{(s^2+0.5858)(s^2+3.414)}{s(s^2+1)(s^2+3)}$.

Theorem 3: $G(s) = \frac{f_n(s)}{l_n(s)}$ represents the transfer function of a controllable and observable loss-less system of order n.

Proof. Let's calculate the zeros Z_h and poles T_h of $G(s)$:

$$\begin{array}{ll} Z_h = 2i\cos\frac{h\pi}{n} \triangleq 2iz_h, & 1 \le h \le n-1 \\ T_h = 2i\cos\frac{(2h-1)\pi}{2n} \triangleq 2it_h, & 1 \le h \le n \end{array} \quad (23)$$

All the poles and zeros of $G(s)$ thus have zero real part. Moreover, they alternate in frequency.

In fact, $z_h < t_h$, since $\frac{h}{n} > \frac{h}{n} - \frac{1}{2n}$, and $t_{h+1} < z_h$, since $\frac{h}{n} + \frac{1}{2n} > \frac{h}{n}$, so that in conclusion:

$$t_1 > z_1 > t_2 > z_2 > \ldots > t_{n-1} > z_{n-1} > t_n$$

which demonstrates that the poles T_h and zeros Z_h of $G(s) = \frac{f_n(s)}{l_n(s)}$ alternate in frequency. ⋄

Example 4: As an example of Theorem 3, consider $G(s) = \frac{f_6(s)}{l_6(s)}$, i.e.

$$G(s) = \frac{s^5 + 4s^3 + 3s}{s^6 + 6s^4 + 9s^2 + 2}$$

$G(s)$ can be factorized as follows:

$$\frac{s(s^2+1)(s^2+3)}{(s^2+0.2679)(s^2+2)(s^2+3.732)}$$

This is the transfer function of a loss-less system. For instance, the pole-zero map shown in Fig. 3 illustrates how the poles and zeros of this system are all on the imaginary axis and alternate. ⋄

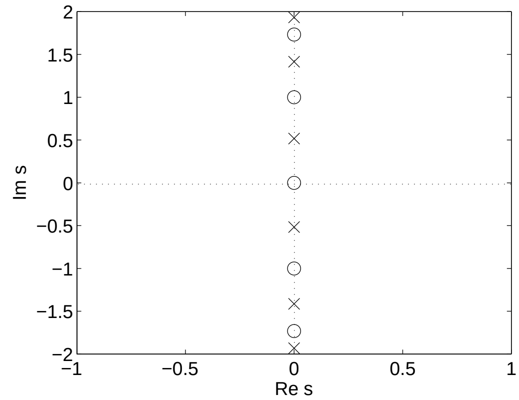

Fig. 3. Pole-zero map of system G(s) in Example 4.

Finally, in the next theorem another way to generate loss-less transfer functions starting from Lucas and Fibonacci polynomials is shown.

Theorem 4: $G(s) = \frac{f_n(s)}{sl_{n-1}(s)}$ represents the transfer function of a loss-less system and the minimal form is of order n, if n is odd, or of order $n-1$, if n is even.

Proof. Analogously to the previously discussed theorems, let's calculate the zeros Z_h and poles T_h of $G(s)$:

$$\begin{array}{rl} Z_h = 2i\cos\frac{h\pi}{n} \triangleq 2iz_h, & 1 \le h \le n-1 \\ T_h = 2i\cos\frac{(2h-1)\pi}{2n} \triangleq 2it_h, & 1 \le h \le n-1 \end{array} \tag{24}$$

Beyond the poles in Eq. (24) another pole $T_n = 0$ should be considered since the denominator of $G(s)$ is given by $sl_{n-1}(s)$.

All the poles and zeros of $G(s)$ have zero real part. Moreover, they alternate in frequency.

In fact, $t_h > z_h$ if $h < \frac{n}{2}$, while $t_h < z_h$ if $h > \frac{n}{2}$ (this can be easily verified taking into account that $\frac{2h-1}{2(n-1)} < \frac{h}{n}$ if $h < \frac{n}{2}$, while $\frac{2h-1}{2(n-1)} > \frac{h}{n}$ if $h > \frac{n}{2}$).

Therefore, one obtains:

$$t_1 > z_1 > t_2 > z_2 > \dots$$

$$\dots > t_{[\frac{n}{2}]} \ge z_{[\frac{n}{2}]} > z_{[\frac{n}{2}]+1} > t_{[\frac{n}{2}]+1} > \dots > z_{n-1} > t_{n-1}$$

Taking also into account the further pole $T_n = 0$, this demonstrates that the poles T_h and zeros Z_h of $G(s) = \frac{f_n(s)}{sl_{n-1}(s)}$ alternate in frequency. The considerations about the order of the minimal form of the system directly follow from the fact that, if n is even, both numerator and denominator polynomial have a root in $s = 0$. ⋄

Example 5: As an example of Theorem 4, consider $G(s) = \frac{f_7(s)}{sl_6(s)}$, i.e.

$$G(s) = \frac{s^6 + 5s^4 + 6s^2 + 1}{s^7 + 6s^5 + 9s^3 + 2s}.$$

$G(s)$ can be factorized as follows:

$$\frac{(s^2 + 0.1981)(s^2 + 1.555)(s^2 + 3.247)}{s(s^2 + 0.2679)(s^2 + 2)(s^2 + 3.732)}$$

This is the transfer function of a loss-less system. For instance, the pole-zero map shown in Fig. 4 illustrates how the poles and zeros of this system are all on the imaginary axis and alternate. ⋄

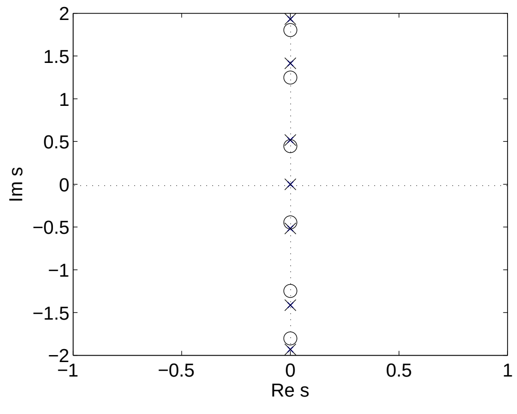

Fig. 4. Pole-zero map of system G(s) in Example 5.

IV. Generating relaxation systems from recursively defined polynomials

Theorem 5: $G_n(s) = \frac{J_n(s)}{J_{n+1}(s)}$ represents the transfer function of a controllable and observable positive real relaxation system of order $[n/2]$.

Proof. The proof proceeds by induction. Consider $n = 4$ (the case with $n < 4$ is trivial), so $G_4(s) = \frac{s+1}{2s+1}$, which is clearly relaxation.

Consider now the following transfer function $F_4(s) = \frac{2s+1}{s(s+1)}$ obtained as $F_n(s) = \frac{J_{n+1}(s)}{sJ_n(s)}$, i.e., $F_n(s) = 1/(sG_n(s))$ with $n = 4$. $F_4(s)$ is clearly a relaxation system, with alternating real negative poles and zeros and with a pole at the origin. We now proceed by induction to prove that, given that $F_{n-1}(s)$ is relaxation with a pole at the origin, also $F_n(s)$ is relaxation with a pole at the origin.

Consider equation (12) with $x = s$ and rewrite it as:

$$\frac{J_n(s)}{sJ_{n-1}(s)} = \frac{1}{s} + \frac{J_{n-2}(s)}{J_{n-1}(s)}$$

i.e.,

$$F_{n-1}(s) = \frac{1}{s} + \frac{1}{sF_{n-2}(s)} \tag{25}$$

Since by hypothesis $F_{n-2}(s)$ is relaxation with a pole at the origin, so $sF_{n-2}(s)$ is relaxation with finite positive values for $s = 0$ and for $s \to \infty$. Therefore, we can consider its inverse (which is relaxation too) and rewrite it in the form of continued fractions as in Eq. (5) as follows:

$$\frac{1}{sF_{n-2}(s)} = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \dots}}$$

Substitute this expression into equation (25) to get:

$$F_{n-1}(s) = \frac{1}{s} + \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \dots}} \tag{26}$$

which is the continued fraction expansion of a positive real relaxation function with a pole at the origin.

Finally, given that $F_n(s)$ is a relaxation transfer function with a pole at the origin, it is immediate to derive that $G_n(s)$ is a positive real relaxation system. The order is given by the polynomial at denominator, i.e., $[n/2]$. ⋄

Example 6: As an example of Theorem 5, consider $G_8(s) = \frac{J_8(s)}{J_9(s)}$, i.e.

$$G_8(s) = \frac{4s^3 + 10s^2 + 6s + 1}{s^4 + 10s^3 + 15s^2 + 7s + 1}$$

$G_8(s)$ can be factorized as follows:

$$G_8(s) = \frac{4(s + 1.707)(s + 0.5)(s + 0.2929)}{(s + 8.291)(s + 1)(s + 0.426)(s + 0.2831)}$$

which is the transfer function of a relaxation system having all the poles and zeros on the real axis and alternating, as also illustrated in Fig. 5 where the pole-zero map of the system is shown. ⋄

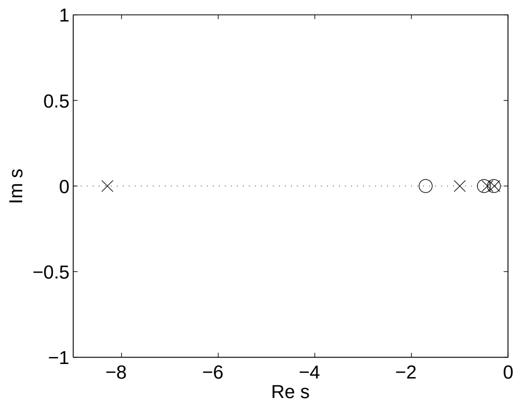

Fig. 5. Pole-zero map of system $G_8(s)$ in Example 6.

Theorem 6: $G_n(s) = \frac{K_n(s)}{K_{n+1}(s)}$ represents the transfer function of a controllable and observable positive real relaxation system of order $[n/2]$.

Proof. The proof is analogous to that of theorem 5 and is therefore omitted. ⋄

Example 7: As an example of Theorem 6, consider $G_7(s) = \frac{K_7(s)}{K_8(s)}$, i.e.

$$G_7(s) = \frac{4s^3 + 7s^2 + 2s}{s^4 + 9s^3 + 9s^2 + 2s}$$

$G_7(s)$ can be factorized as follows:

$$G_7(s) = \frac{4s(s + 1.39)(s + 0.3596)}{s(s + 7.892)(s + 0.7858)(s + 0.3225)}$$

which is the transfer function of a relaxation system having all the poles and zeros on the real axis and alternating, as also shown in Fig. 6. ⋄

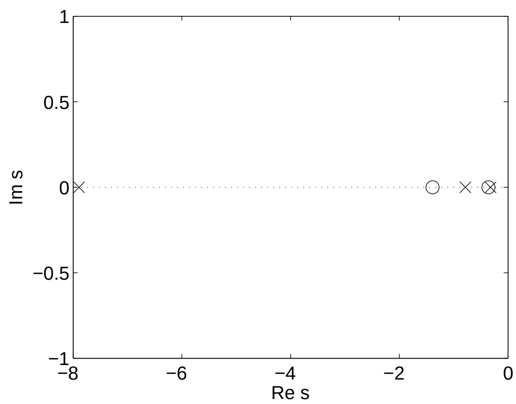

Fig. 6. Pole-zero map of system $G_7(s)$ in Example 7.

We now examine Morgan-Voyce polynomials. Also these polynomials can be used to generate transfer functions of relaxation systems.

Theorem 7: $G_n(s) = \frac{b_n(s)}{b_{n+1}(s)}$ represents the transfer function of a controllable and observable positive real relaxation system of order $n + 1$.

Proof. We prove that $G_n(s) = \frac{b_n(s)}{b_{n+1}(s)}$ is the transfer function of a positive real relaxation system by showing that its zeros and poles are real, negative and alternate in frequency. The zeros L_h and the poles M_h of $G_n(s)$ are the zeros of $b_n(s)$ and $b_{n+1}(s)$, respectively. So, thanks to property 6, they can be explicitly calculated:

$$\begin{aligned} L_h &= -4\sin^2 \tfrac{(2h-1)\pi}{(4n+2)}, \quad 0 \le h \le n-1 \\ M_h &= -4\sin^2 \tfrac{(2h-1)\pi}{(4n+6)}, \quad 0 \le h \le n \end{aligned} \tag{27}$$

Note that $\frac{(2h-1)\pi}{(4n+2)} \in [0, \pi/2]$ and $\frac{(2h-1)\pi}{(4n+6)} \in [0, \pi/2]$. Since in the interval $[0, \pi/2]$, $\sin^2 x$ is a monotonically increasing function and $\frac{(2h-1)\pi}{(4n+2)} > \frac{(2h-1)\pi}{(4n+6)}$ for $0 \le h \le n-1$, then

$$L_h < M_h \tag{28}$$

for $0 \le h \le n-1$.

Moreover, since $\frac{(2n-3)\pi}{(4n+2)} < \frac{(2n-1)\pi}{(4n+6)}$, then

$$L_{n-1} > M_n \tag{29}$$

From Eqs. (28) and (29) it follows that

$$M_1 > L_1 > M_2 > L_2 > \ldots > M_{n-1} > L_{n-1} > M_n$$

which complete the proof. ⋄

Example 8: As an example of Theorem 7, consider $G_3(s) = \frac{b_3(s)}{b_4(s)}$, i.e.

$$\frac{s^3 + 5s^2 + 6s + 1}{s^4 + 7s^3 + 15s^2 + 10s + 1} \tag{30}$$

$G_3(s)$ can be factorized as follows:

$$\frac{(s + 3.247)(s + 1.555)(s + 0.1981)}{(s + 3.532)(s + 2.347)(s + 1)(s + 0.1206)} \tag{31}$$

This is the transfer function of a positive real relaxation system. For instance, the pole-zero map shown in Fig. 7 illustrates how the poles and zeros of this system are all on the real axis and alternate. ⋄

Theorem 8: $G_n(s) = \frac{B_n(s)}{B_{n+1}(s)}$ represents the transfer function of a controllable and observable positive real relaxation system of order $n + 1$.

Proof. The proof of this theorem is similar to that of theorem 7. In fact, it is based on showing by direct calculation that the zeros and poles of this transfer function are real, negative and alternate in frequency. Let the zeros and the poles of $G_n(s) = \frac{B_n(s)}{B_{n+1}(s)}$ be indicated as U_h and V_h, respectively. Thanks to property 7, they can be explicitly calculated:

$$\begin{aligned} U_h &= -4\sin^2 \tfrac{h\pi}{(2n+2)}, \quad 0 \le h \le n-1 \\ V_h &= -4\sin^2 \tfrac{h\pi}{(2n+4)}, \quad 0 \le h \le n \end{aligned} \tag{32}$$

It is immediate to derive that: i) $\frac{h\pi}{(2n+2)} \in [0, \pi/2]$ and $\frac{h\pi}{(4n+6)} \in [0, \pi/2]$; ii) $\frac{h\pi}{(2n+2)} > \frac{h\pi}{(2n+4)}$ for $0 \leq h \leq n-1$; iii) $\frac{n\pi}{(2n+2)} < \frac{(n+1)\pi}{(2n+4)}$. Therefore, it follows that

$$V_1 > U_1 > V_2 > U_2 > \ldots > V_{n-1} > U_{n-1} > V_n$$

which complete the proof. ◇

Example 9: As an example of Theorem 8, consider $G_8(s) = \frac{B_8(s)}{B_9(s)}$, i.e.

$$G_8(s) = \frac{s^8 + 16s^7 + 105s^6 + 364s^5 + 715s^4 + 792s^3 + 462s^2 + 120s + 9}{s^9 + 18s^8 + 136s^7 + 560s^6 + 1365s^5 + 2002s^4 + 1716s^3 + 792s^2 + 165s + 10} \tag{33}$$

$G_8(s)$ can be factorized as follows:

$$G_8(s) = \frac{(s+3.879)(s+3.532)(s+3)(s+2.347)(s+1.653)(s+1)(s+0.4679)(s+0.1206)}{(s+3.902)(s+3.618)(s+3.176)(s+2.618)(s+2)(s+1.382)(s+0.8244)(s+0.382)(s+0.09789)} \tag{34}$$

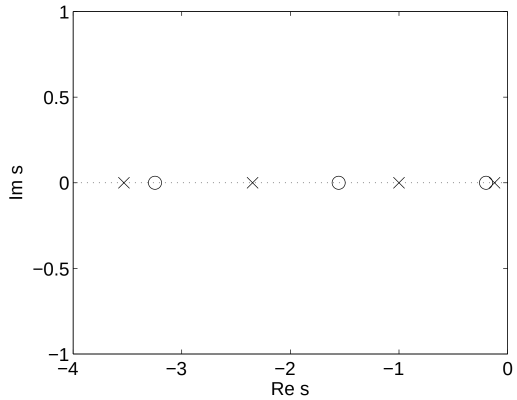

Fig. 7. Pole-zero map of system $G_3(s)$ in Example 8.

This is the transfer function of a positive real relaxation system. For instance, the pole-zero map shown in Fig. 8 illustrates how the poles and zeros of this system are all on the real axis and alternate. ◇

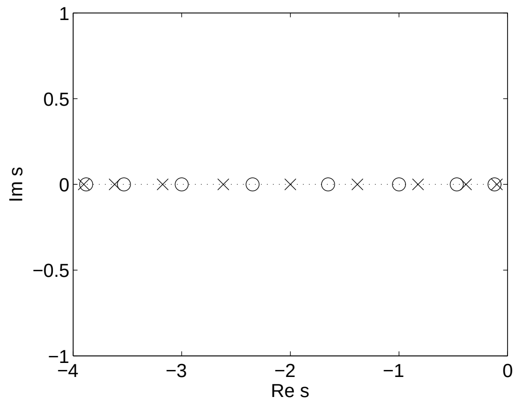

Fig. 8. Pole-zero map of system $G_8(s)$ in Example 9.

Finally, let us remark that not all recursively defined polynomials generate loss-less or relaxation transfer functions as discussed in the following example.

Example 10: Let us consider polynomials defined by:

$$D_n(s) = D_{n-1}(s) + sD_{n-2}(s) \tag{35}$$

with $D_1 = s^2 + 2s + 2$ and $D_2 = s^2 + s + 1$. Eq. (35) is analogous to Eq. (12), but they differ for the initial polynomials.

Consider then $D_3(s) = s^3 + 3s^2 + 3s + 1$ and $D_4(s) = 2s^3 + 4s^2 + 4s + 1$ and the transfer function defined as $G(s) = \frac{D_3(s)}{D_4(s)} = \frac{s^3+3s^2+3s+1}{2s^3+4s^2+4s+1}$. This is neither loss-less or relaxation. In fact, it can be rewritten as:

$$G(s) = \frac{0.5(s+1)^3}{(s+0.3522)(s^2+1.648s+1.42)}$$

which shows that the zeros and poles do not satisfy the conditions required for the system to be loss-less or relaxation. ◇

V. Conclusions

Fibonacci, Lucas, Jacobsthal, Morgan-Voyce polynomials are all sequences of polynomials recursively defined, which have several interesting properties. In this paper it has been shown that, thanks to their characteristics, SISO LTI systems with peculiar properties can be defined. In fact, it has been demonstrated that single-input single-output linear time-invariant systems having as transfer function the ratio between two successive Fibonacci polynomials, the ratio between two successive Lucas polynomials, the ratio between a Fibonacci polynomial of order n and a Lucas polynomial of order n or the ratio between a Fibonacci polynomial of order n and a Lucas polynomial of order $n-1$ multiplied by the complex variable s, are loss-less. Furthermore, it has been shown that, when Jacobsthal or Morgan-Voyce polynomials are considered, passive relaxation systems are obtained.

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TABLE V

LIST OF THE FIRST TEN MORGAN-VOYCE POLYNOMIALS $b(x)$

Order	Morgan-Voyce polynomials $b(x)$
$n = 0$	$b_0(x) = 1$
$n = 1$	$b_1(x) = x + 1$
$n = 2$	$b_2(x) = x^2 + 3x + 1$
$n = 3$	$b_3(x) = x^3 + 5x^2 + 6x + 1$
$n = 4$	$b_4(x) = x^4 + 7x^3 + 15x^2 + 10x + 1$
$n = 5$	$b_5(x) = x^5 + 9x^4 + 28x^3 + 35x^2 + 15x + 1$
$n = 6$	$b_6(x) = x^6 + 11x^5 + 45x^4 + 84x^3 + 70x^2 + 21x + 1$
$n = 7$	$b_7(x) = x^7 + 13x^6 + 66x^5 + 165x^4 + 210x^3 + 126x^2 + 28x + 1$
$n = 8$	$b_8(x) = x^8 + 15x^7 + 91x^6 + 286x^5 + 495x^4 + 462x^3 + 210x^2 + 36x + 1$
$n = 9$	$b_9(x) = x^9 + 17x^8 + 120x^7 + 455x^6 + 1001x^5 + 1287x^4 + 924x^3 + 330x^2 + 45x + 1$

TABLE VI

LIST OF THE FIRST TEN MORGAN-VOYCE POLYNOMIALS $B(x)$

Order	Morgan-Voyce polynomials $B(x)$
$n = 0$	$B_0(x) = 1$
$n = 1$	$B_1(x) = x + 2$
$n = 2$	$B_2(x) = x^2 + 4x + 3$
$n = 3$	$B_3(x) = x^3 + 6x^2 + 10x + 4$
$n = 4$	$B_4(x) = x^4 + 8x^3 + 21x^2 + 20x + 5$
$n = 5$	$B_5(x) = x^5 + 10x^4 + 36x^3 + 56x^2 + 35x + 6$
$n = 6$	$B_6(x) = x^6 + 12x^5 + 55x^4 + 120x^3 + 126x^2 + 56x + 7$
$n = 7$	$B_7(x) = x^7 + 14x^6 + 78x^5 + 220x^4 + 330x^3 + 252x^2 + 84x + 8$
$n = 8$	$B_8(x) = x^8 + 16x^7 + 105x^6 + 364x^5 + 715x^4 + 792x^3 + 462x^2 + 120x + 9$
$n = 9$	$B_9(x) = x^9 + 18x^8 + 136x^7 + 560x^6 + 1365x^5 + 2002x^4 + 1716x^3 + 792x^2 + 165x + 10$