Sampling Procedure of the Arithmetic Operations with two Binary Markov Process Realizations

Y. Goritskiy, V. Kazakov

Abstract— The sampling procedure of the arithmetic operation with two Binary Markov Processes (BMP) is considered in detail, especially for the sum of BMP. The result of the sum can have four or three states. In the first case there is a Markovian process, in the second case the process is non Markovian. We investigate sampling procedure of any realization of this case. The investigation method takes into account the probability of the state omission and probability densities functions of staying times in each state. We obtain the algorithm for choosing of sampling intervals for each state. They are different. One non trivial example is given. The obtained results are generalized for other arithmetic operations: the multiplication and division. One rather important variant of the application of such processes in the random field model is given at the final paragraph.

Keywords— Sampling Algorithm, Binary Markov Process, Arithmetic Operations with two Binary Markov Processes.

I. Introduction

THE problem of sampling - reconstruction procedure (SRP) of realizations of random processes is extensively examined in literature. Majority of publications are devoted to the SRP investigation of realizations of *continuous* processes. SRP of realizations of *discontinuous* processes is weakly considered. Here we mention some papers connected with the discussed problem [1, 2]. In [1] sampling of realizations of the Binary Markov Process (BMP) is investigated in order to describe new Binary processes. In [2] SRP of realizations of a function of BMP is analyzed. But this analysis is based on the covariance approach only. There are some authors publications [3 - 7] devoted to SRP of *discontinuous* process realizations. Investigations in [5 - 7] are based on the Conditional mean Rule. In results we obtained the statistical description of sampling procedure and to estimate the q quality of the reconstruction. It is necessary to mention that there is a great difference between the SRP methods for continuous [8 – 11] and discontinuous [5 - 7, 12, 13] process realizations.

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In the present paper we consider SRP of realizations of discontinuous process which is the arithmetic operation with two independent BMP processes. Such a problem is interesting in some control and telemetry systems. There are two different cases: 1) the sampled process has four different states; 2) the process has three different states. The first variant can be considered by the Markov method suggested in [5 - 7]. The investigation of the second case is more difficult in comparison with the first one because the process under consideration becomes non Markov. Here a new aspect of SRP problem is occurred. In the first time, we need to investigate the determination of the instant moment of the next sample, when the locations of previous samples are known. This problem must be solved taking into account the given probability of a state omission. This SRP problem is investigated in detail. Finally, there is one non trivial example.

II. The studied variant

For the sake of simplicity, we consider the variant of sum of two BMP realizations in detail. At the end of paper other arithmetic operations will be discussed.

Let $\xi_1(t)$ and $\xi_2(t)$ are independent BMP with corresponding states: $\{x_0, x_1\}$ and $\{y_0, y_1\}$, $x_0 < x_1$, $y_0 < y_1$, and with transition densities (or with the intensities) $\{\lambda_0, \lambda_1\}$ and $\{\mu_0, \mu_1\}$. These densities have a known sense, for instance, λ_0:

$P\{x_0 \to x_1, \Delta t\} = \lambda_0 \Delta t + o(\Delta t)$ etc.

Let us introduce a two dimensional Markov process $\{\xi_1(t), \xi_2(t)\}$ with four states. The process graph is presented in Fig. 1. Here one can see states with all possible transitions and corresponding densities.

We consider a sum

$$\xi(t) = \xi_1(t) + \xi_2(t) .$$

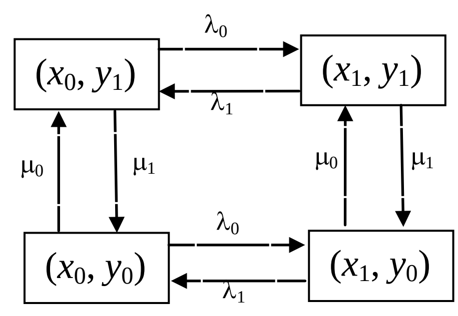

Fig. 1 Graph of the two dimensional process $\{\xi_1(t), \xi_2(t)\}$

The process $\xi(t)$ is observable. We have to obtain a sampling algorithm for any realization of this process with the principal condition: the probability of a state omission must be no more of a given value γ.
Let us consider two different cases.

A. The First Case

When

$$\Delta x \equiv x_1 - x_0 \neq \Delta y \equiv y_1 - y_0; \qquad (1)$$

the process $\xi(t)$ has *four* states z_0, z_1, z_2, z_3:

$$z_0 = x_0 + y_0,\ z_1 = x_0 + y_1,$$
$$z_2 = x_1 + y_0,\quad z_3 = x_1 + y_1$$

Since $z_1 \neq z_2$, taking into account (1) we have $z_2 - z_1 = (x_1 - x_0) - (y_1 - y_0) \neq 0$. Then the process $\xi(t)$ is Markovian. The graph of this process is given in Fig. 2. In this case the probability density function (pdf) of staying time η_i in the states $z_i (i = 0,1,2,3)$ is exponential $E(\alpha_i)$ with parameters:

$$\eta_0 \sim E(\alpha_0 = \lambda_0 \cdot + \mu_0),\qquad \eta_1 \sim E(\alpha_1 = \lambda_0 \cdot + \mu_1),$$
$$\eta_2 \sim E(\alpha_2 = \lambda_1 \cdot + \mu_0),\qquad \eta_3 \sim E(\alpha_3 = \lambda_1 \cdot + \mu_1) \qquad (2)$$

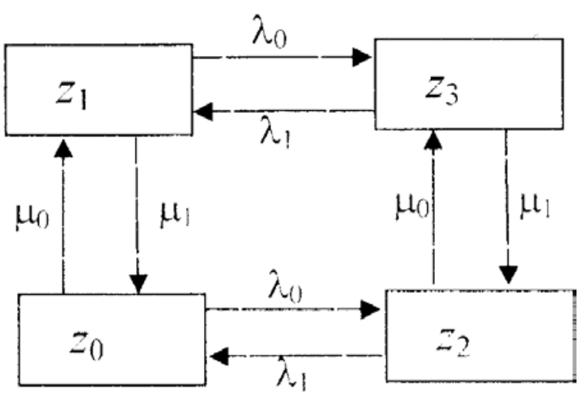

Fig. 2. Graph of transitions of the $\xi(t)$ process for the first case

The transition probabilities are determined by the following formulas:

$$P_{01} = \frac{\mu_0}{\mu_0 + \lambda_0},\quad P_{02} = \frac{\lambda_0}{\mu_0 + \lambda_0},$$
$$P_{31} = \frac{\lambda_1}{\lambda_1 + \mu_1},\quad P_{32} = \frac{\mu_1}{\lambda_1 + \mu_1} \qquad (3)$$

In (3) we give some required expressions only. In order to describe SRP of such type of the process, it is necessary to use the method [5 - 7].

B. The Second Case

When

$$\Delta x \equiv x_1 - x_0 = \Delta y \equiv y_1 - y_0; \qquad (4)$$

the process $\xi(t)$ has *three* states z_0, u, z_3:

$$z_0 = x_0 + y_0,\ u = x_0 + y_1, = x_1 + y_0,\ z_3 = x_1 + y_1$$

The state u is formed by the union of two states z_1 and z_2 of the process in the first case. Fig. 3 illustrates the expression (4): the line of the constant sum passes through two states of the two dimensional process. The graph of transitions of the process $\xi(t)$ is shown in Fig. 4.

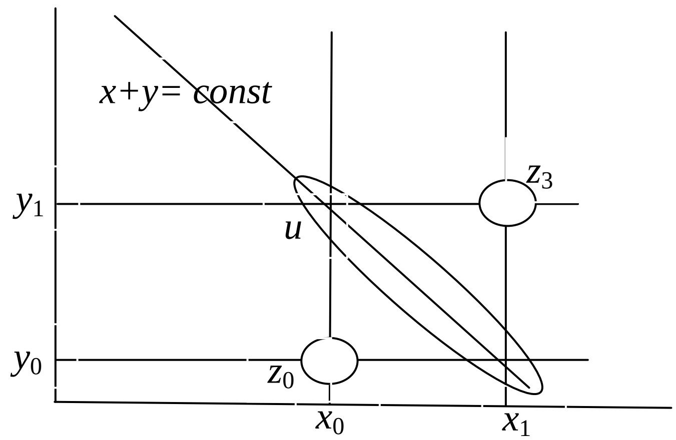

Fig.3. Graph of the process $\xi(t)$ for the second case

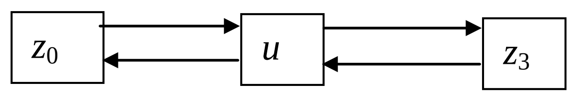

Fig.4. Graph of the transitions of the process $\xi(t)$ for the second case

Let us discuss the situation with the time states. The pdf of time staying η_0 and η_3 in the states z_0 and z_3 are the same as in (2), i.e. they are exponential with the parameters α_0 and α_3 correspondingly. The situation with pdf of staying time η_u in the state u is another. The pdf of the staying time η_u depends on the previous state $z_i\,(i = 0,3)$. If the realization comes into the state u from z_0, then the pdf of the staying time η_u will be determined by the formula of the total probability

$$f_u(t|0) = p_{01}E(t,\alpha_1) + p_{02}E(t,\alpha_2) \qquad (5a)$$

If the realization comes into the state u from z_3, then pdf of the staying time η_u will be determined by another formula of the total probability

$$f_u(t|3) = p_{31}E(t,\alpha_1) + p_{32}E(t,\alpha_2) \qquad (5b)$$

It means that pdf of staying time is not exponential. In this case it is necessary to apply analysis which is valid for non Markovian processes, or for an arbitrary pdf.

III. The Estimation of the Sampling Interval

Let us introduce some designations (Fig. 5):
i, j, k – the past, the presence, and the future states.

t_L - the last time moment of the measurement of the state $\xi(t_L) = i$ and $\xi(t_L + T_i) = j \neq i$. This means that the change of the state occurs in the interval $(t_L, t_L + T_i)$.

τ_{ij} - the random time interval from t_L until the time moment of the change state $i \to j$.

$w_{ij}(x), 0 < x < T_i,$ - the corresponding pdf of the random time interval.

η_j - the random staying time in the state j; it is described by the pdf $f_j(t)$ and by the distribution function $F_j(t)$.

ς_j - a random time interval of the output from the state j; this interval has its initial value at the moment t_L:

$$\varsigma_j = \tau_{ij} + \eta_j \tag{6}$$

$h_j(t)$ - the pdf of the random variable ς_j; this pdf is expressed by the formula

$$h_j(t) = \int_0^{\min(t,T_j)} f_j(t-x) w_{ij}(x) dx, 0 < t < \infty \tag{7}$$

Below we shall investigate the value $t \geq T_i$; then the upper limit in (7) will be T_i.

Let us assume that there were some samples (it is possible one sample) and t_p is the time moment in the present, i.e. $\xi(t_p) = j$.

$\theta = t_p - t_L$ - is an interval from t_L until a current present time t_p, $\theta \geq T_i$.

$\zeta_{\theta j}$ – a remainder of θ: a waiting time for output from j if $\zeta_j > \theta$. It means that there was not any output from j during the interval, i.e.

$$\varsigma_{\theta j} = (\varsigma_j | \varsigma_j > \theta) - \theta$$

The corresponding pdf is determined by

$$h_{\theta j} = c_1 h_j(\theta + t), 0 < t < \infty \tag{8}$$

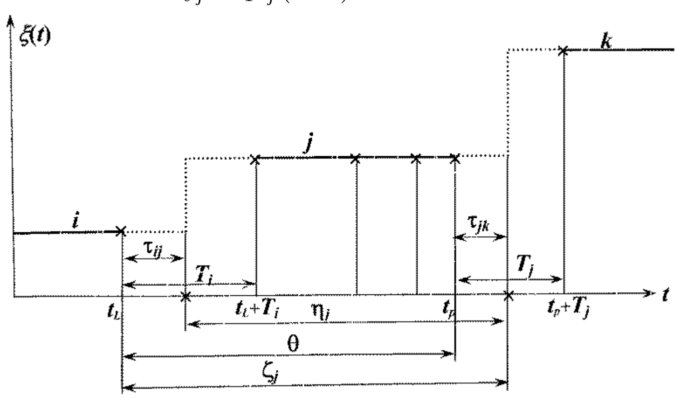

Fig. 5. The illustration of designations

Here c_1 is a normalizing coefficient:

$$c_1 = \int_0^{\infty} h_j(\theta + t) dt) = R_{\varsigma j}(\theta) = \int_{\theta}^{\infty} h_j(t) dt \tag{9}$$

The sampling interval T_j can be chosen taking into account the required demand for the probability of the omission of the next state k:

$$\begin{aligned} &\max_k \mathrm{P}\{\varsigma_{\theta j} + \eta_k < T_j\} = \mathrm{P}\{\varsigma_{\theta j} + \eta_{k^*} < T_j\} = \\ &= \mathrm{P}\{\Sigma < T_j\} = F_{\Sigma}(T_j; j, k^*, \theta) = \gamma \end{aligned} \tag{10}$$

where $F_{\Sigma}(t; j, k, \theta)$ – is the probability distribution function of the sum

$$\Sigma \equiv \zeta_{\theta j} + \eta_k \tag{11}$$

k^*- is a number of state which gives the minimal value of the interval T_j.

Taking into account the expression (8) and the independence of the items, the distribution function $F_{\Sigma}(t; j, k, \theta)$ is determined by the integral:

$$\begin{aligned} &F_{\Sigma}(t; j, k, \theta) = c_1 \int_0^t h_j(\theta + x) F_k(t - x) dx = \\ &= c_1 \int_{\theta}^{\theta + t} h_j(x) F_k(\theta + t - x) dx \end{aligned} \tag{12}$$

Let us consider two different cases: the random staying time is described by: A) the exponential law and B) the non exponential law.

A. The PDF of the Staying Time is Exponential

If a current state j is equal to z_0 or z_3 (they have the exponential pdf), the remainders of waiting states $\varsigma_{\theta j}, j = 0;3$ have the same pdf and they do not depend on θ.

Then we have $f_j(t) = E(t, \alpha_j)$, j = 0; 3. The staying time η_u depends on j. Its pdf is determined by the combination of two exponential pdf, corresponding to states z_1 and z_2. Pdf for η_u is described by the expression (see, (5a) and (5b)):

$$f_u(t|j) = \sum_{s=1}^{2} p_{js} E(t; \alpha_s), \quad = 0; \quad 3. \tag{13}$$

Let us designate:

$$R(t|j) = e^{-\alpha t}, F(t; \alpha) = 1 - e^{-\alpha t} \tag{14}$$

and then

$$R_u(t|j) = P\{\eta_u(j) > t\} = \sum_{s=1}^{2} p_{js} R(t; \alpha_s)$$

Let us introduce intensities λ_1 and λ_2 of two exponential laws. The distribution function of the sum of two independent exponential random variables can be simply found by the formula:

$$G(t; \lambda_1, \lambda_2) = 1 - \frac{\lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} - \frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}, if\, \lambda_1 \neq \lambda_2 \tag{15}$$

$$G(t; \lambda_1, \lambda_2) = 1 - e^{-\lambda_1 t}(1 + \lambda_1 t), if\, \lambda_1 = \lambda_2 \tag{16}$$

Then the distribution function $F_{\Sigma}(t; z_j, u)$ for the expression (13) is written in the form:

$$F_{\Sigma}(t; z_j, u) = F_{\Sigma}(t; z_j, u) = \sum_{s=1}^{2} p_{js} G(t; \alpha_j, \alpha_s), j\text{=0;3} \tag{17}$$

Thus, taking into account (10), (15) – (17), sampling intervals T_0 and T_3 can be determined from the equations:

$$F_\Sigma\left(T_j; j,u\right)=\gamma,\ j=0;3 \qquad (18)$$

B. The PDF of the Staying Time is not Exponential

Let us assume the current state j is the state u. Then the sampling interval depends on: 1) the previous state (z_0 or z_3), 2) the staying time θ in the current state u , 3) the future state, which gives the minimal value of the interval.

Below pdf $w_{ij}\left(x\right)\left(0<x<T_i\right)$ plays an important part. So, we make a note about this pdf: the distribution of a transition moment in the next interval can be found on the basis of the distribution in the previous interval. Therefore we can assume that pdf $w_{ij}\left(x\right)$ is known in time t_L+T_i .

Now we determine the distribution $w_{zu}\left(t\right)$ for the time moment τ_{zu} of the transition from z_i into *u*.

Taking into account results [5, 6] and the expression (13), one can obtain the following formula for pdf $w_{zu}\left(t\right)=w_{iu}\left(t\right), i=0;3$ as a linear combination of two exponential truncated pdf. The argument of this pdf is restricted by the value T_i. We designate this pdf by $E_{Ti}\left(t;\mu_{is}\right)$ ($\mu_{is}=\alpha_i-\alpha_s, i=0;3, s=1;2$)

$$E_{Ti}\left(t;\mu_{is}\right)=\begin{cases}\mu_{is}e^{-\mu_{is}t}\left(1-e^{-\mu_{is}T_i}\right)^{-1}, if\ \mu_{is}\neq 0\\ 1/T_i, if\ \mu_{is}=0\end{cases} \qquad (19)$$

The expression for the required pdf has the form:

$$w_{iu}\left(t\right)=\sum_{s=1}^{2}p_{is}E_{Ti}\left(t;\mu_{is}\right), i=0;3, s=1;2. \qquad (20)$$

On the basis of (20) one can find some initial and central moments.

The next step of our investigation is connected with the determination of pdf for the sum $\Sigma=\varsigma_{\theta u}+\eta_k$.

Pdf of the output time moment ς_u can be found by (7) with the change $j\to u$. The calculations are illustrated by the following formulas:

$$h_u\left(t|i\right)=\int_0^{T_i}\sum_{s=1}^{2}p_{is}E\left(t-x;\alpha_s\right)\sum_{r=1}^{2}p_{ir}E_{Ti}\left(x;\mu_{ir}\right)dx=$$

$$=\sum_{r=1}^{2}\sum_{s=1}^{2}p_{is}p_{ir}\int_0^{T_i}E\left(t-x;\alpha_s\right)E_{Ti}\left(x;\mu_{ir}\right)dx=$$

$$=\sum_{r=1}^{2}\sum_{s=1}^{2}p_{is}p_{ir}d_{irs}\alpha_s e^{-\alpha_s t}=\sum_{s=1}^{2}q_{is}\alpha_s e^{-\alpha_s t}, i=0;3. \qquad (21)$$

where

$$d_{irs}=\alpha_s e^{\alpha_s t}\int_0^{T_i}E\left(t-x;\alpha_s\right)E_{Ti}\left(x;\mu_{ir}\right)dx=$$

$$=\begin{cases}\dfrac{\mu_{ir}\left(1-e^{-\left(\mu_{ir}-\alpha_s\right)T_i}\right)}{\left(\mu_{ir}-\alpha_s\right)\left(1-e^{-\mu_{ir}T_i}\right)}, if\ \mu_{ir}\neq 0, \mu_{ir}-\alpha_s\neq 0,\\ \dfrac{\mu_{ir}T_i}{\left(1-e^{-\mu_{ir}T_i}\right)}, if\ \mu_{ir}\neq 0, \mu_{ir}-\alpha_s=0,\\ \dfrac{e^{\alpha_s T_i}-1}{\alpha_s T_i}, if\ \mu_{ir}=0\end{cases} \qquad (22)$$

$$q_{is}=p_{is}\sum_{r=1}^{2}p_{ir}d_{irs}, i=0;3, q=1,2. \qquad (23)$$

We note that d_{irs} involves the information about the previous transition. Let us determine some other functions.

$$R_{\varsigma u}\left(t|i\right)=P\left\{\varsigma_u>t|i\right\}=\int_t^{\infty}h_u\left(x|i\right)dx=$$

$$=\sum_{s=1}^{2}q_{is}e^{-\alpha_s t}, i=0;3. \qquad (24)$$

Here ς_u is a remainder of a waiting time of the output from the state u with the condition $\varsigma_u>\theta$:

$$\varsigma_{\theta u}=\left(\varsigma_u\left|\varsigma_u>\theta\right.\right)-\theta$$

The corresponding pdf will be:

$$h_{\theta u}=c_2h_u\left(\theta+t|i\right), 0<t<\infty \qquad (25)$$

where c_2 is the normalizing coefficient:

$$c_2^{-1}=\int_0^{\infty}h_u\left(\theta+t|i\right)dt=P\left\{\varsigma_u>\theta\right\}=$$

$$=R_{\varsigma u}\left(\theta\right)=\sum_{s=1}^{2}q_{is}e^{-\alpha_s\theta} \qquad (26)$$

The next sampling interval T_u will be determined from the expression (10). In order to do this, we find the principal expression for the function $F_\Sigma\left(T;u,k,\theta\right)$:

$$F_\Sigma\left(T;u,k,\theta\right)=P\left(\varsigma_{\theta u}+\eta_k<t|i\right)=\int_0^t P\left(\eta_k<t-x\right)\times$$

$$\times h_{\theta u}\left(x|i\right)dx=c_2\int_0^t h_u\left(\theta+x|i\right)F_k\left(t-x\right)dx=$$

$$=c_2\int_0^t\sum_{s=1}^{2}q_{is}\alpha_s e^{-\alpha_s\left(\theta+x\right)}\left(1-e^{-\alpha_k\left(t-x\right)}\right)dx=$$

$$=c_2\int_0^t h_u\left(\theta+x\right)dx-c_2\sum_{s=1}^{2}q_{is}\alpha_s e^{-\alpha_s\theta}e^{-\alpha_k t}\times$$

$$\times\int_0^t e^{-\nu_{sk}x}dx=c_2\left[R_{\varsigma u}\left(\theta|i\right)-R_{\varsigma u}\left(\theta+t|i\right)\right]-$$

$$-c_2\sum_{s=1}^{2}\sum_{s=1}^{2}q_{is}\alpha_s e^{-\alpha_s\theta}e^{-\alpha_k t}\times\begin{cases}\dfrac{1-e^{-t\nu_{sk}}}{\nu_{sk}}, if\ \nu_{sk}\neq 0\\ t, if\ \nu_{sk}=0\end{cases}=$$

$$= c_2 \sum_{s=1}^{2} q_{is} e^{-\alpha_s \theta}\left(1-e^{-\alpha_s t}\right) - c_2 \sum_{s=1}^{2} q_{is} \alpha_s e^{-\alpha_s \theta} \times$$

$$\times \begin{cases} \dfrac{e^{-\alpha_k t} - e^{-\alpha_s t}}{\nu_{sk}}, if\, \nu_{sk} \neq 0 \\ e^{-\alpha_k t}, if\, \nu_{sk} = 0 \end{cases} = \quad (27)$$

$$= \frac{\sum_{s=1}^{2} q_{is} e^{-\alpha_s \theta}\left[\left(1-e^{-\alpha_s t}\right) - b_{sk}(t)\right]}{\sum_{s=1}^{2} q_{is} e^{-\alpha_s \theta} \times},$$

$$b_{sk} = \begin{cases} \dfrac{\alpha_k}{\mu_{sk}}\left(e^{-\alpha_k t} - e^{-\alpha_s t}\right), if\, \mu_{sk} \neq 0 \\ \alpha_k t e^{-\alpha_k t}, if\, \mu_{sk} = 0 \end{cases} \quad (28)$$

$$\mu_{sk} = \alpha_s - \alpha_k$$

For these cases one can determine sampling intervals $T_u(\theta, i), i = 0;3$ following the equation

$$F_{\Sigma}\left(T_u; u, k^*, \theta \middle| i\right) = \gamma \quad (29)$$

In order to find the reconstruction algorithm it is necessary to determine pdf for the transition time moment τ_{uk}. Following [13], one can write

$w_{uk}(t) = c_3 h_{\theta u}(t|i) R_k(T_u - t), 0 < t < T_u, T_u = T_u(\theta, i)$ Using expressions (21), (25), we obtain

$$w_{uk}(t) = c_3 \sum_{s=1}^{2} q_{is} \alpha_s e^{-\alpha_s \theta} e^{-\mu_{sk} t}, 0 < t < T_u \quad (30)$$

$(i, k = 0;3)$.

The normalizing coefficient c_3 is determined by

$$c_3^{-1} = \int_0^{T_u} h_{\theta u}(t|i) R_k(T_u - t)\, dt = \sum_{s=1}^{2} q_{is} \alpha_s e^{-\alpha_s \theta} \times$$

$$\times \begin{cases} \dfrac{1-e^{-\mu_{sk} T_u}}{\mu_{sk}}, if\, \mu_{sk} \neq 0 \\ T_u, if\, \mu_{sk} = 0 \end{cases} \quad (31)$$

On the basis of (30) and (31) one can find the expression for two initial momemts $m_l(u,k), l = 1;2$:

$$m_l(u,k) = \int_0^{T_u} t^l w_{uk}(t)\, dt = c_3 \sum_{s=1}^{2} q_{is} \alpha_s e^{-\alpha_s \theta} \times$$

$$\times \begin{cases} \dfrac{l!}{\mu_{sk}^{\,l+1}}\left[1 - e^{-\mu_{sk} T_u} \sum_{i=0}^{l} \dfrac{\left(\mu_{sk} T_u\right)}{i!}\right], if\, \mu_{sk} \neq 0 \\ \dfrac{T_u^{\,l+1}}{l+1}, if\, \mu_{sk} = 0, k = 0;3, l = 1;2 \end{cases} \quad (32)$$

IV. Example

In order to illustrate the influence of staying interval in the state u on the SRP, it would be better to choose different values of α_1 and α_2 (for instance, $\alpha_1 = k\alpha_2, k \gg 1$). Besides this, it is better to choose $p_{31} = 1-\varepsilon_1, p_{02} = 1-\varepsilon_2$; $\varepsilon_1, \varepsilon_2 \ll 1$. Then the intensities of the initial processes can be written:

$\lambda_0 = k(1-\varepsilon_2)h, \mu_0 = k\varepsilon_2 h, \lambda_1 = (1-\varepsilon_1)h, \mu_1 = \varepsilon_1 h$

here h is a positive constant. Let us put:

$k = 5, \varepsilon_1 = \varepsilon_2 = 0.1, h = 2c$, then we have

$p_{01} = 0.1, p_{31} = 0.9, p_{02} = 0.9, p_{31} = 0.1,$

$\alpha_1 = 9.2c, \alpha_2 = 2.8c, \alpha_0 = 10c, \alpha_3 = 2c$

Putting c=1/56 we obtain the following values of staying times:

T_{z0} = 5.6, T_{z1} = 6.1, T_{z2} = 20, T_{z3} =28.

The average staying time in the state u depends on the previous state: if the process realization comes into z_u from z_0, then $T_u(z_0) = 18.6$; if the process realization comes from z_3 then $T_u(z_3) = 7.4$. In Fig. 6 two conditional pdf $f_u(t|z_i), i = 0;3$ are presented.

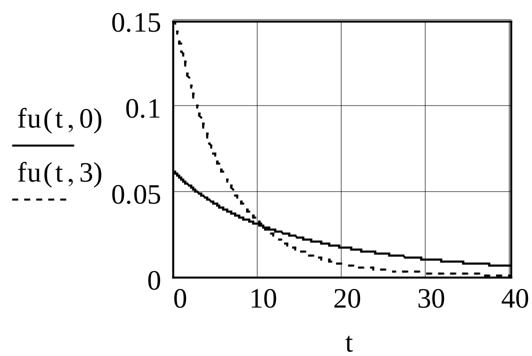

Fig. 6. The conditional pdf $f\left(\eta_u \middle| z_i\right)$ for two values z_i (i=0; 3).

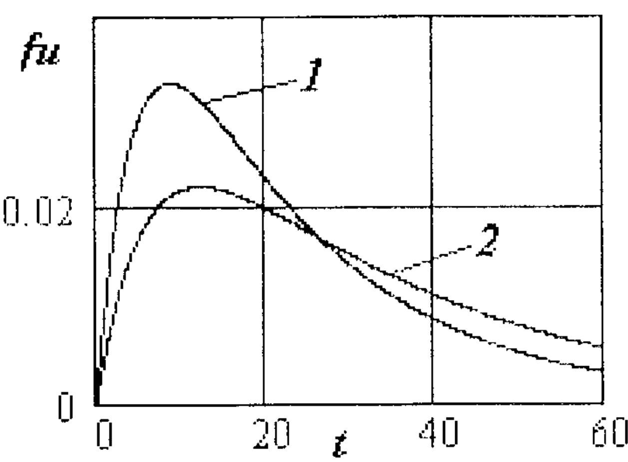

Fig. 7a. Pdf of the sum (11).

As one can see the distribution of the staying time in the state u are different depended on the previous state: the curve 1 corresponds to the state z_0; the curve 2 corresponds to the state z_3.

In Fig. 7a two pdf (13) for the sum (11) are shown. These curves correspond to states z_0 and z_3. Fig.7b illustrates the initial parts of the corresponded distribution functions (10). The graphs of the time length $T_u(\theta, i, k^*)$ in the state u as a function of the staying time θ with a different previous states *i*=0; 3 are given in Fig. 8. The worst future state is $k^* = 0$ because $T_0 < T_3$.

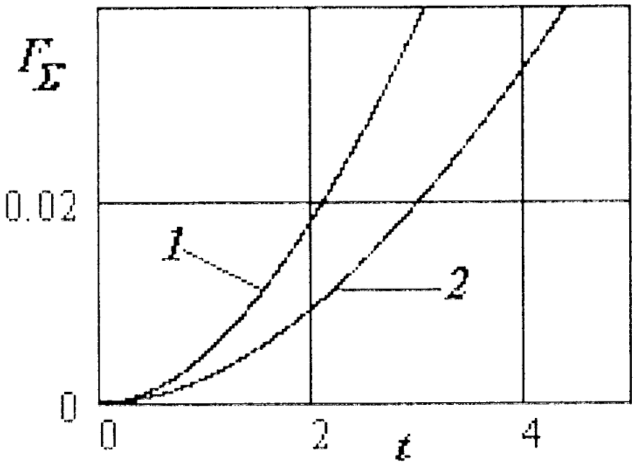

Fig. 7b. Distribution functions of the sum (11).

Let us choose the required probability of a state omission $\gamma = 0.02$. The investigated realization $\xi(t)$ is the consequence of states z_0=0, *u*=1.5, z_3=3, *u*=1.5, z_3=3. We put the initial value $\xi(t_0 = 0) = 0$. The concrete corresponding random staying times are equal: 5.1, 15, 21.1, 5.2.

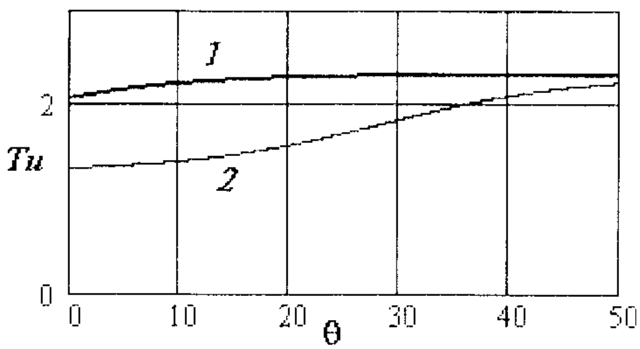

Fig. 8. Time length $T_u(\theta, i, k^*)$ in the state *u*.

Calculations of the sampling intervals T_i and $T_u(z_i)$ are characterized by the following results:

1) $T_0 = 2.1$; $T_3 = 3.02$. 2) $T_u(z_0)$=2.11, 2.15, 2.18, 2.21, 2.23, 2.25, 2.26; $T_u(z_3)$=1.34, 1.35, 1.36.

In Fig 9, one realization of the process $\xi(t)$ is presented. Besides this, there are some samples designated by crosses. As one can see, the sampling intervals $\Delta(z_0)$ and $\Delta(z_3)$ are constant and they do not changed in time because both pdf of the staying times are exponential. But the intervals are different, because they depend on pdf of staying times in considered and future states. The intervals in the state *u* are different as well and they are changed in time. However this dependence is rather small. This effect is determined by previous state z_0 or z_3 and by the conditional pdf of the staying times.

In order to describe reconstruction points and to estimate the reconstruction error it is necessary to determine the conditional pdf of the jump instant time between two neighbor samples.

V. Other Arithmetic Operations

One can investigate SRP of realizations which form by a multiplication and a division of two realizations of independent BMP. The general situation is the same like we have in the sum operation.

In the case of the multiplication we consider

$$\xi(t) = \xi_1(t)\xi_2(t) \tag{33}$$

If $x_0 y_1 \neq x_1 y_0$, the process $\xi(t)$ has four states and this process is Markovian. If $x_0 y_1 = x_1 y_0$, the process $\xi(t)$ has three states and this process is non Markovian.

In the case of the division we consider

$$\xi(t) = \xi_1(t) / \xi_2(t) \tag{34}$$

The condition $x_0 y_0 \neq x_1 y_1$ means that the process $\xi(t)$ has four states and this process is Markovian. If $x_0 y_0 = x_1 y_1$, the process $\xi(t)$ has three states and this process is non Markovian.

The investigation method for the Markovian case is described in n. 2 and in [5, 6]. The non Markovian case is the main content of the presented paper.

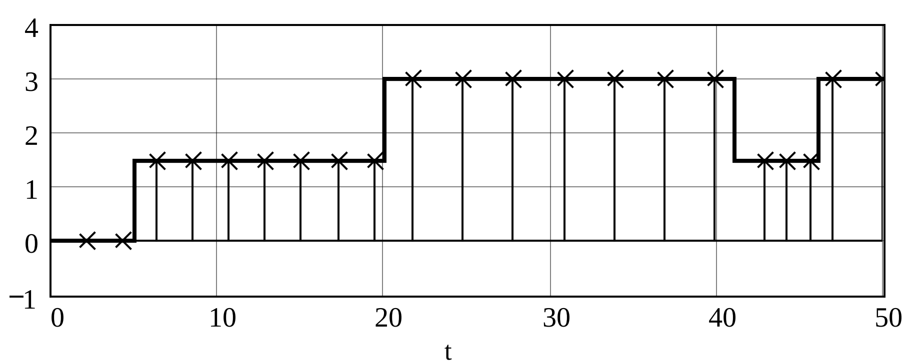

Fig. 9. One realization of the sum process and the set of samples

VI. About the Application of Obtained Results

The above considered process formed by the arithmetic operations can be applied in the description of the mathematical model of random fields with jumps [14]. This field is formed in following manner: we fix two realizations of different BMP along each axis. A brightness of any point of a field can be determined by the arithmetic function $z = g(\xi_1, \xi_2)$ of BMP realizations. This field is characterized by four (or three) states. It is clear that this field is non Gaussian. It is possible to find the principal characteristic of such field – its space covariance function. Let us restrict by the case of four numbers of states. Then the general expression for the space covariance function is determined by the formula:

$$K(\Delta x, \Delta y) = \sum_{x_i, x_l'} \sum_{y_j, y_k'} g(x_i, y_j) g(x_l', y_k') \times P(x_i, y_j; x_l', y_k'; \Delta x, \Delta y) - \left[\sum_{x_i, y_j} g(x_i, y_j) P(x_i, y_j)\right]^2 \quad (35)$$

where

Δx, Δy are the distances between x_i, x_l' and y_j, y_k' correspondingly; $P(\bullet)$ is the probability of correspondent random variables.

Because $\xi_1(t)$ and $\xi_1(t)$ are independent, it is sufficiently to know the probabilities $P(x_i, x_l, \Delta x), P(y_j, y_k, \Delta y)$, $P(x_i)$, and $P(y_j)$. These probabilities can be found from the solution of the system of Kolmogorov´s differential equations for the Markovian chains with continuous time. We have following results for the process $\xi_1(t)$:

$$P(x_0, x_0', \Delta x) = \frac{\lambda_1^2}{\lambda_x^2}\left(1 - e^{-\lambda_x|\Delta x|}\right) + \frac{\lambda_1}{\lambda_x} e^{-\lambda_x|\Delta x|} \quad (36)$$

$$P(x_0, x_1', \Delta x) = P(x_1, x_0', \Delta x) = \frac{\lambda_0 \lambda_1}{\lambda_x^2}\left(1 - e^{-\lambda_x|\Delta x|}\right) \quad (37)$$

$$P(x_1, x_1', \Delta x) = \frac{\lambda_0^2}{\lambda_x^2}\left(1 - e^{-\lambda_x|\Delta x|}\right) + \frac{\lambda_0}{\lambda_x} e^{-\lambda_x|\Delta x|} \quad (38)$$

where $\lambda_x = \lambda_0 + \lambda_1$

Putting $\Delta x = 0$, from (36) – (38) we obtain

$$P(x_1) = \frac{\lambda_1}{\lambda_x}, P(x_0) = \frac{\lambda_0}{\lambda_x} \quad (39)$$

The corresponding expressions for probabilities of the process $\xi_2(t)$ can be written by the change letters “λ” into “μ” with corresponding indexes.

Taking into account (35) – (39), one can obtain the general expression for the space covariance function:

$$K(\Delta x, \Delta y) = (\lambda_x \mu_y)^{-2} \Big\{ \left[g^2(x_0, y_0) + g^2(x_0, y_1)\right] \times \left(e^{-\mu_y|\Delta y|} \lambda_1^2 \mu_0 \mu_1 + e^{-\lambda_x|\Delta x|} \lambda_1 \lambda_0 \mu_1^2 + \gamma_{xy}\right) + \left[g^2(x_1, y_0) + g^2(x_1, y_1)\right]\left(\ddot{e}^{-\mu_y|\Delta y|} \lambda_0^2 \mu_0 \mu_1 + \gamma_{xy}\right) +$$
$$+ 2g(x_0, y_0) g(x_0, y_1)\left[-e^{-\mu_y|\Delta y|} \lambda_1^2 \mu_1 \mu_0 + \gamma_{xy}\left(e^{\mu_y|\Delta y|} - 1\right)\right] + + 2g(x_1, y_0) g(x_0, y_0)\left[-e^{-\lambda_x|\Delta x|} \lambda_0 \lambda_1 \mu_1^2 + \gamma_{xy}\left(e^{\lambda_x|\Delta x|} - 1\right)\right] +$$
$$+ 2\left[g(x_0, y_0) g(x_1, y_1) + g(x_0, y_1) g(x_1, y_0)\right] \gamma_{xy} \times \left(1 - e^{\mu_y|\Delta y|} - e^{\lambda_x|\Delta x|}\right) + 2g(x_0, y_1) g(x_1, y_1)\left[-e^{-\lambda_x|\Delta x|} \mu_1^2 \lambda_0 \lambda_1 + \right.$$
$$\left. + 2g(x_1, y_0) g(x_1, y_1)\left[e^{-\mu_y|\Delta y|} \lambda_1^2 \mu_0 \mu_1 + \gamma_{xy}\left(e^{-\mu_y|\Delta y|} - 1\right)\right] \Big\} \quad (40)$$

where $\gamma_{xy} = \lambda_0 \lambda_1 \mu_0 \mu_1 e^{-\lambda_x|\Delta x| - \mu_y|\Delta y|}$

The formula (40) is principal for the calculations. Below we give the final expressions of the space covariance functions for all arithmetic operations.

1) $\xi(t) = \xi_1(t) \pm \xi_2(t)$

$$K(\Delta x, \Delta y) = \frac{\lambda_0 \lambda_1}{\lambda_x^2}(x_0 - x_1)^2 e^{-\lambda_x|\Delta x|} + \frac{\mu_0 \mu_1}{\mu_x^2}(y_0 - y_1)^2 e^{-\mu_y|\Delta y|} \quad (41)$$

2) $\xi(t) = \xi_1(t) \xi_2(t)$

$$K(\Delta x, \Delta y) = (\lambda_x \mu_y)^{-2}\Big[\mu_0 \mu_1 (y_0 - y_1)^2 (x_1 \lambda_0 + x_0 \lambda_1)^2 e^{-\mu_y|\Delta y|} + \lambda_0 \lambda_1 (x_1 - x_0)^2 (y_0 \mu_1 + y_1 \mu_0)^2 e^{-\lambda_x|\Delta x|} + \lambda_0 \lambda_1 \mu_0 \mu_1 \times (y_1 - y_0)^2 (x_1 - x_0)^2 e^{-\lambda_x|\Delta x| - \mu_y|\Delta y|}\Big] \quad (42)$$

3) $\xi(t) = \xi_1(t) / \xi_2(t)$

$$K(\Delta x, \Delta y) = (\lambda_x \mu_y)^{-2}\Big[\mu_0 \mu_1 (x_0 \lambda_1 + x_1 \lambda_0)^2 \left(\frac{1}{y_0} - \frac{1}{y_1}\right)^2 \times e^{-\mu_y|\Delta y|} + \lambda_0 \lambda_1 \left(\frac{\mu_1}{y_0} + \frac{\mu_0}{y_1}\right)(x_1 - x_0)^2 e^{-\lambda_x|\Delta x|} + \lambda_0 \lambda_1 \mu_0 \mu_1 \times \left(\frac{1}{y_0} - \frac{1}{y_1}\right)^2 (x_1 - x_0)^2 e^{-\lambda_x|\Delta x|} e^{-\mu_y|\Delta y|}\Big] \quad (43)$$

One can find the expressions for the space power spectra $S(\omega_x, \omega_y)$, which is determined by the formula

$$S(\omega_x, \omega_y) = \int\int K(\Delta x, \Delta y) e^{-(\omega_x \Delta x + \omega_y \Delta y)} d(\Delta x) d(\Delta y) \quad (44)$$

Let us specify the formula (44) for the covariance function (42):

$$S(\omega_x, \omega_y) = (\lambda_x \mu_y)^{-2}\Big[\lambda_0 \lambda_1 (y_0 - y_1)^2 (x_1 \lambda_0 + x_0 \lambda_1) \times \frac{2\mu_y}{\mu_y^2 + \omega_y^2} + \lambda_0 \lambda_1 (x_1 - x_0)^2 (y_0 \mu_1 + y_1 \mu_0)^2 \frac{2\lambda_x}{\lambda_x^2 + \omega_x^2} + \lambda_0 \lambda_1 \mu_0 \mu_1 (x_1 - x_0)^2 (y_1 - y_0)^2 \frac{4\lambda_x \mu_y}{(\lambda_x^2 + \omega_x^2)(\mu_y^2 + \omega_y^2)}$$

It is quite possible to obtain corresponding expressions for other variants of covariance functions.

We note that mathematical models found on the basis of some arithmetic operation with BMP provide a great possibility to change their features owing many parameters. The SRP of random processes and random fields on the basis of BMP can be described using results of the present paper.

VII. Conclusion

In the first time a sampling problem of special realizations is solved. These realizations are formed by the arithmetic operation with two binary Markovian processes. The Markovian and non Markovian cases are investigated. The algorithms for choosing intervals are obtained. One rather important variant of the application of such processes in the random field model is given.

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