Introduction to the Angular Coordinate System in 2D space

Claude Ziad Bayeh

Abstract--- The Angular Coordinate System is a new and original mathematical Coordinate System introduced by the author in the mathematical domain. It has own properties similar to other coordinate systems such as Cylindrical coordinate system, Cartesian coordinate system or any other coordinate system. The main goal of introducing this coordinate system is to describe the whole universe in 2D space using two angles only formed by the center of a unit circle of the universe. This new coordinate system is introduced also to facilitate the drawing of many complicated curves that are difficult to produce using the traditional coordinate systems. It has many applications in physics for example the unit circle can be considered as a black hole and we have to describe the formed universe according to this black hole using only two angles. In this paper a complete study is introduced for the new coordinate system and few examples are presented in order to give an idea about how to form and study curves in the Angular coordinate system.

Key-words--- Angular coordinate system, two angles, unit circle, black hole, Cartesian coordinate system, mathematics.

I. INTRODUCTION

IN mathematics, there exist numerous coordinate systems in different spaces such as Cartesian coordinate system [1] [2], cylindrical coordinate system [3], Polar coordinate system [4], Bipolar coordinates [5], Elliptic coordinate system [6] and many others coordinate systems [7-11]. The main goal of introducing these systems was to facilitate the study of certain curves in each system and each system has its own properties and definitions that allow it to form an independent coordinate system. The most important coordinate system and the easiest one is the Cartesian coordinate system in which one can easily determine any point in the space by two coordinates (x, y). Some curves are easiest to be determined in some coordinates rather than others, for this reason we see a huge number of coordinates system that can be very useful in some cases. And also for this reason, the author introduced a new coordinate system named Angular coordinate system.

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In mathematics, the Angular coordinate System is a new and original coordinate system introduced by the author in the mathematical domain. The main goal of this new coordinate system is to construct a new space based on two angles only. These two angles (α_1, α_2) can determine the position of any point in the Angular coordinate system 2D space similar to the Cartesian coordinate system in which every point has two coordinates $(X_p;\ Y_p)$.

If we talk about the Cartesian coordinate system in 2D space, every point (P) has two coordinates, the first is horizontal coordinate (X_p) which determine the position of the point (P) in the horizontal axis, the second is the vertical coordinate (Y_p) which determine the position of the point (P) in the vertical axis. The same principal can be applied on the Angular Coordinate system in 2D Space; every point has two coordinates (α_1, α_2), the first one (α_1) is the coordinate of the point (P) from the center of the space (also called center of the unit circle), this angle is determined in degree or in radian (it will be explained in the following sections), the straight line that passes from the center of the space (the center of the unit circle) and passes by the point (P) form an angle with the horizontal axis called (α_1). The second coordinate of the point (P) is also an angle named (α_2) which is the angle of the formed tangent on the unit circle that also passes by the point.

The two formed angles create a space measured only by these two angles instead of the linear coordinate system. So every point in the space has two angles (α_1, α_2), every curve in the space depend on two angles which are also (α_1, α_2) so we can create curves by using these two angles for example $\alpha_1 = 2\alpha_2 + \cos(\alpha_2)$.

In the second section, a definition of the unit circle is presented. In the section 3, some applications in mathematics of the Angular coordinate system are presented. In the section 4, Properties of the Angular coordinate system in 2D space are presented. In the section 5, Steps of study for a function in the angular coordinate system are presented. In the section 6, examples of applications are presented. And finally in the section 7, a conclusion is presented.

II. DEFINITION OF THE UNIT CIRCLE

The unit circle (refer to figure 1) is the center of the space formed by two angles (α_1, α_2), the formed space is outside the unit circle, which means we can't find any point inside the circle. This can have many applications in physics to describe the universe and how the space is formed outside the black holes and stars. The radius of the unit circle is equal to "1" in this study but it can take any value depending on the usage for example it can take the radius of the Sun if we want to study the objects that turn around it, it can take the radius of a black hole in order to study the objects around it and so on.

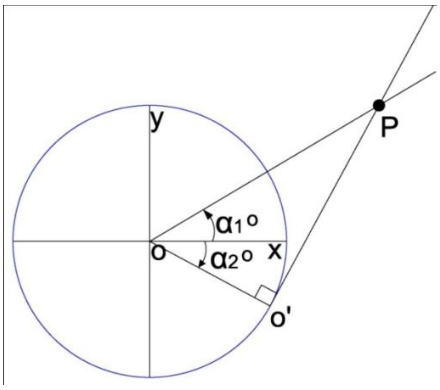


Fig.1: represents the unit circle with two angles (α_1, α_2) and the point P outside the circle.

In the figure 1, the straight line formed by the center of the unit circle (O) and the point (P) outside the circle is called "center line", this line has an angle (α_1) from the axis "OX", the formed angle helps us to determine the position of the point (P) in the space.

The line (O'P) formed by the point (O') which is tangent on the unit circle and the point (P) in the space named "tangent line" forms a segment (OO') with an angle (α_2) from the axis (OX).

The intersection of the "center line" and the "tangent line" form the point (P). So every point in the space is formed by the intersection of two lines "center line" and "tangent line", which form two angles (α_1, α_2). Therefore every point in the space is determined by two unique angles (α_1, α_2). The curves can be drawn by using one angle in function of the other one, for example $\alpha_1 = 2\alpha_2 + \sqrt{(\alpha_2 + 5)}$.

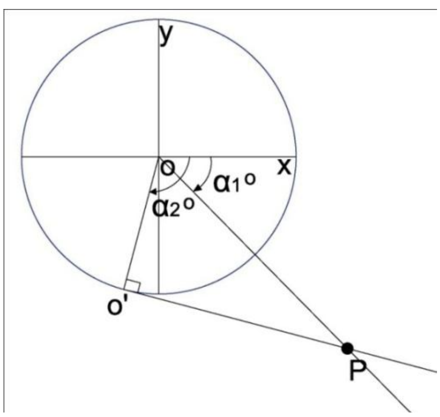


Fig.2: represents the unit circle with two angles (α_1, α_2) and the point P outside the circle.

The angle $(\widehat{OO'P})$ is always equal to 90^0 as it forms the tangent line on the circle that passes on the point (P) and the point (O').

The angles (α_1) and (α_2) are always measured from the (OX) axis.

The angle $(\widehat{O'OP}) = \alpha_1 - \alpha_2$.

Therefore, $\sin(\alpha_1 - \alpha_2) = \frac{O'P}{OP}$
And $\cos(\alpha_1 - \alpha_2) = \frac{OO'}{OP} = \frac{R}{OP}$

So the distance $OP = \frac{R}{\cos(\alpha_1 - \alpha_2)}$ (1)
the radius of the unit circle is known (equal to 1) so it is easy to determine the distance (OP) by knowing the two angles (α_1, α_2).

The angle α_1 is always greater or equal to the angle α_2.
If $\alpha_1 = \alpha_2$ then the point (P) is on the circle.

$0 \leq \alpha_1 - \alpha_2 < 90^0$ (2)

A. Instrument made for Angular coordinate system
We can make an Angular coordinate system as an instrument used to determine the distance of any point in the space by following these steps:
- make a plastic plate as circular piece and put the angles on it as in the figure 3.
- put two heads of laser, the first one is on the "center line" and the other one is on the "tangent line" as in figure 4.
-in order to determine the distance of a point from the center we should turn the lasers on and center the lights of lasers to pass through the point (P) as in figure 5.
-in the figure 5, we can read the angles (α_1) and (α_2) then determine the position of the point (P) and the distance (OP). Therefore $OP = \frac{R}{\cos(\alpha_1 - \alpha_2)}$.
If we take the figure 5, we consider that R=0.5m, and we read that $\alpha_1 = 26.2^0$ and $\alpha_2 = -38^0$ therefore $OP = \frac{R}{\cos(\alpha_1 - \alpha_2)} = \frac{0.5}{\cos(26.2 - (-38))} = 1.148m$ and the point (P) has the following coordinates $P(\alpha_1; \alpha_2) = P(26.2^0; -38^0)$

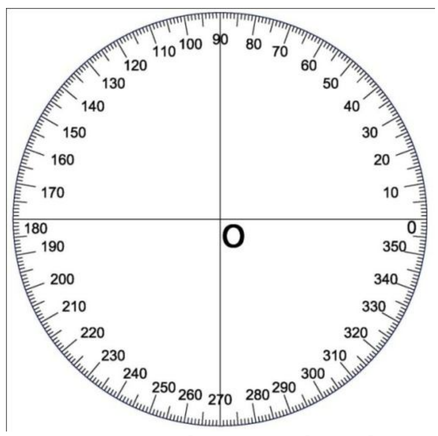


Fig.3: represents the unit circle made as an instrument.

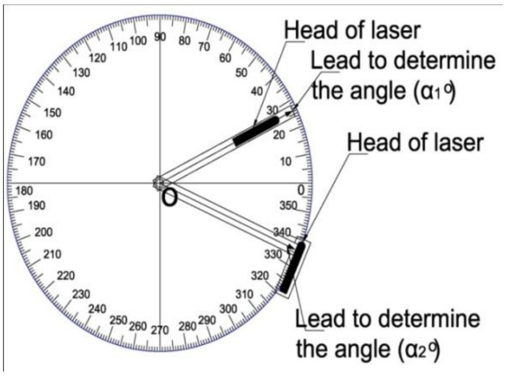


Fig.4: represents the unit circle made as an instrument with two heads of lasers one on the "center line" and the other on the "tangent line".

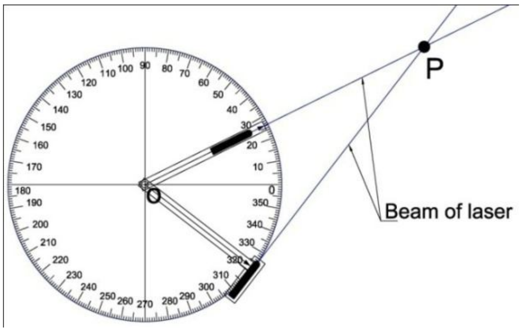


Fig.5: represents how one can determine the position of a point in the space by the intersection of the beams of laser at the point (P).

III. APPLICATION IN MATHEMATICS OF THE ANGULAR COORDINATE SYSTEM

A. Distance between two points M and N in the 2D space

Let's consider two points in the space $M(\alpha_1;\alpha_2)$ and $N(\beta_1;\beta_2)$ (refer to figure 6).

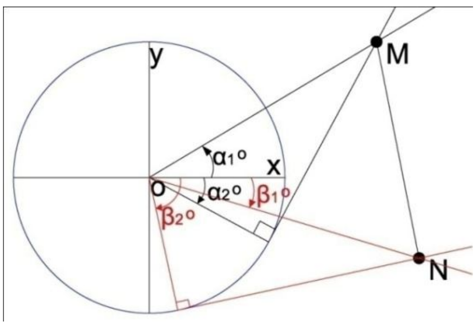


Fig.6: represents the distance between two points M and N.

We can determine the distance between the two points using the angular coordinate system as the following:

$$OM = \frac{R}{\cos(\alpha_1 - \alpha_2)} \quad (3)$$

and $ON = \frac{R}{\cos(\beta_1 - \beta_2)}$ (4)

The angle $\widehat{MON} = \widehat{MOX} + \widehat{XON} = \alpha_1 - \beta_1$ (5)

And $\frac{\sin(\widehat{MON})}{MN} = \frac{\sin(\widehat{OMN})}{ON} = \frac{\sin(\widehat{ONM})}{OM}$ (6)

The equation (6) is the relation of angles and segments inside the triangle.

$$MN^2 = OM^2 + ON^2 - 2 \cdot OM \cdot ON \cdot \cos(\widehat{MON}) \quad (7)$$

$$\Rightarrow MN^2 = OM^2 + ON^2 - 2 \cdot OM \cdot ON \cdot \cos(\alpha_1 - \beta_1) \quad (8)$$

We replace the equations (3) and (4) in the equation (8) and we obtain

$$MN^2 = \left(\frac{R}{\cos(\alpha_1 - \alpha_2)}\right)^2 + \left(\frac{R}{\cos(\beta_1 - \beta_2)}\right)^2 - 2R^2 \cdot \frac{\cos(\alpha_1 - \beta_1)}{\cos(\alpha_1 - \alpha_2)\cos(\beta_1 - \beta_2)}$$

Considering that

$\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$

Therefore:

$$MN^2 = \left(\frac{R}{\cos(\alpha)}\right)^2 + \left(\frac{R}{\cos(\beta)}\right)^2 - 2R^2 \cdot \frac{\cos(\alpha_1 - \beta_1)}{\cos(\alpha)\cos(\beta)}$$

$$MN = R\sqrt{\frac{1}{\cos(\alpha)^2} + \frac{1}{\cos(\beta)^2} - 2\frac{\cos(\alpha_1 - \beta_1)}{\cos(\alpha)\cos(\beta)}} \quad (9)$$

The equation (9) is the equation of the distance between two points in the Angular coordinate system 2D space.

B. Equation of a line in the Angular coordinate system in 2D space

Considering the equation of a line in the Cartesian coordinate system such as $y = ax + b$. We have to convert this equation into the Angular coordinate system depending on angles. This can be done by using the following steps:

Let's consider a point (M) on the line, then $M \in y$ and $y_M = ax_M + b$

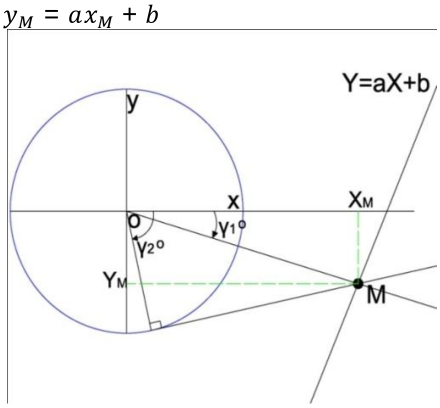


Fig.7: represents the line $y = ax + b$ and M is a point of this line.

Considering the coordinate of the point (M) in the Angular coordinate system is $M(\gamma_1;\gamma_2)$

With $\gamma_1 = \widehat{XOM}$

Therefore, $\sin(\gamma_1) = \frac{y_M}{OM} \Rightarrow y_M = OM \cdot \sin(\gamma_1)$ (10)

$\cos(\gamma_1) = \frac{x_M}{OM} \Rightarrow x_M = OM \cdot \cos(\gamma_1)$ (11)

By replacing the equations (10) and (12) in the equation $y_M = ax_M + b$ we obtain:

$$OM \cdot \sin(\gamma_1) = a \cdot OM \cdot \cos(\gamma_1) + b \quad (13)$$

We have to determine the parameters a and b. So we consider two point also included into the line which are $A(\alpha_1;\alpha_2)$ and $B(\beta_1;\beta_2)$. The coordinate of the two point verify the equation of the line so we can write the following equations:

$$\begin{cases} OA \cdot \sin(\alpha_1) = a \cdot OA \cdot \cos(\alpha_1) + b & (14) \\ OB \cdot \sin(\beta_1) = a \cdot OB \cdot \cos(\beta_1) + b & (15) \end{cases}$$

To find the parameter a we subtract (15)-(14) and we obtain:

$$a = \frac{OB\cdot\sin(\beta_1) - OA\cdot\sin(\alpha_1)}{OB\cdot\cos(\beta_1) - OA\cdot\cos(\alpha_1)} \quad (16)$$

In order to eliminate the "center lines" OA and OB we use the equation (3)

$$OA = \frac{R}{\cos(\alpha_1-\alpha_2)} = \frac{R}{\cos(\alpha)} \quad (17)$$

$$OB = \frac{R}{\cos(\beta_1-\beta_2)} = \frac{R}{\cos(\beta)} \quad (18)$$

We replace the equation (17) and (18) in the equation (16) and we obtain:

$$a = \frac{\cos(\alpha)\cdot\sin(\beta_1) - \cos(\beta)\cdot\sin(\alpha_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} \quad (19)$$

The equation (19) is the equation of the parameter a

To find the parameter b we replace the equation (19) in the equation (14) and we obtain:

$$b = R\frac{\cos(\beta_1)\cdot\sin(\alpha_1) - \cos(\alpha_1)\cdot\sin(\beta_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} \quad (20)$$

The following important formula (21) can be used

$$\sin(m-n) = \cos(n)\sin(m) - \cos(m)\sin(n) \quad (21)$$

So, $b = R\frac{\sin(\alpha_1-\beta_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)}$ (22)

The equation (22) is the equation of the parameter b

The equations (10), (11), (19) and (22) can be replaced in the equation $y_M = ax_M + b$ and we obtain,

$$\sin(\gamma_1) = \cos(\gamma_1)\frac{\cos(\alpha)\sin(\beta_1) - \sin(\alpha_1)\cos(\beta)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} + \cos(\gamma)\frac{\sin(\alpha_1-\beta_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} \quad (23)$$

Considering that

$$a_1 = \frac{\cos(\alpha)\sin(\beta_1) - \sin(\alpha_1)\cos(\beta)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} = a \quad (24)$$

$$b_1 = \frac{\sin(\alpha_1-\beta_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} = \frac{b}{R} \quad (25)$$

Therefore the equation (23) will be as the following:

$$\sin(\gamma_1) = \cos(\gamma_1)\, a_1 + \cos(\gamma) b_1 \quad (26)$$

The equation (26) is the equation of a line in the Angular coordinate system.

The equation (26) can be simplified as the following:

$$\frac{\sin(\gamma_1)}{\cos(\gamma_1)} = a_1 + \frac{\cos(\gamma_1)\cos(\gamma_2) + \sin(\gamma_1)\sin(\gamma_2)}{\cos(\gamma_1)} b_1$$

$$\Rightarrow \tan(\gamma_1) = a_1 + b_1(\cos(\gamma_2) + \tan(\gamma_1)\sin(\gamma_2))$$

$$\Rightarrow \gamma_1 = tan^{-1}\left(\frac{a_1 + b_1\cos(\gamma_2)}{1 - b_1\sin(\gamma_2)}\right) \quad (27)$$

The equation (27) is the simplified equation of a line in the Angular coordinate system.

B.1. Example of an equation of a line in the Angular coordinate system in 2D space

Considering the equation of a line in the Cartesian coordinate system such as $y = x - 8$.

We have to convert this equation into the Angular coordinate system depending on angles. This can be done by using the following steps:

• Suppose that the point Q is included in the line and $Q(5,-3)$, therefore the distance

$$OQ = \sqrt{5^2 + 3^2} = \sqrt{34} = 5.83095$$

$$\alpha_1 = \begin{cases} \pm\arccos\left(\frac{x_Q}{\sqrt{(x_Q)^2 + (y_Q)^2}}\right) \\ \pm\arcsin\left(\frac{y_Q}{\sqrt{(x_Q)^2 + (y_Q)^2}}\right) \end{cases} = \begin{cases} 30.9637^0 \\ -30.9637^0 \end{cases}$$

$$\Rightarrow \alpha_1 = -30.9637^0$$

And

$$\alpha_2 = \alpha_1 - \arccos\left(\frac{R}{\sqrt{(x_Q)^2 + (y_Q)^2}}\right) = -111.0887^0$$

Therefore $Q(-30.9637^0, -111.0887^0)$ is the coordinates of the point Q in the Angular coordinate system

• Suppose that the point P is included in the line and $P(-2,-10)$, therefore the distance

$$OP = \sqrt{2^2 + 10^2} = \sqrt{104} = 10.198$$

$$\beta_1 = \begin{cases} \pm\arccos\left(\frac{x_p}{\sqrt{(x_P)^2 + (y_P)^2}}\right) \\ \pm\arcsin\left(\frac{y_P}{\sqrt{(x_P)^2 + (y_P)^2}}\right) \end{cases} = \begin{cases} 101.3099^0 \\ -78.69^0 \end{cases}$$

$$\Rightarrow \beta_1 = -101.3099^0$$

And

$$\beta_2 = \beta_1 - \arccos\left(\frac{R}{\sqrt{(x_P)^2 + (y_P)^2}}\right) = -185.6825^0$$

Therefore $P(-101.3099^0, -185.6825^0)$ is the coordinates of the point P in the Angular coordinate system.

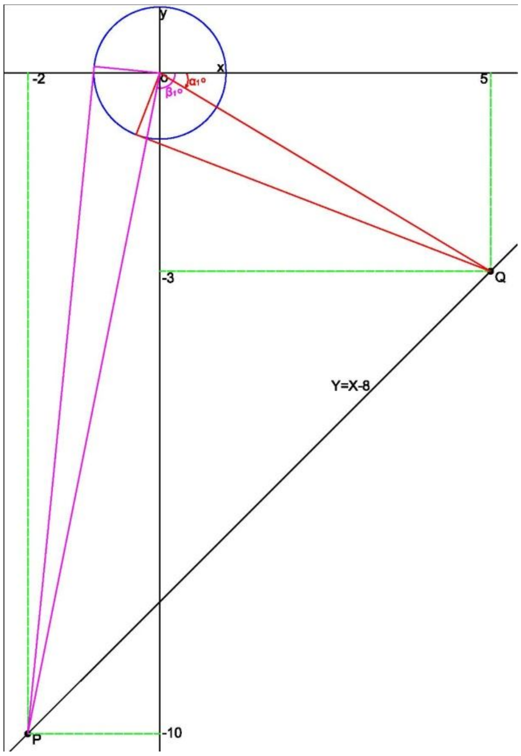


Fig.8: represents the line $y = x - 8$ and two points Q and P of this line.

By knowing the coordinates of two points P and Q in the Angular coordinate system it is easily to determine the equation of the line in the Angular coordinate system.

$$a_1 = \frac{\cos(\alpha)\sin(\beta_1) - \sin(\alpha_1)\cos(\beta)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} \quad (24)$$

$$a_1 = \frac{\cos(80.125^0)\sin(-101.3099^0) - \sin(-30.9637^0)\cos(84.3726^0)}{\cos(80.125^0)\cdot\cos(-101.3099^0) - \cos(84.3726^0)\cdot\cos(-30.9637^0)}$$

$$\Rightarrow a_1 = 1$$

$$b_1 = \frac{\sin(\alpha_1 - \beta_1)}{\cos(\alpha)\cdot\cos(\beta_1) - \cos(\beta)\cdot\cos(\alpha_1)} \quad (25)$$

$$b_1 = \frac{\sin(-30.9637^0 + 101.3099^0)}{\cos(80.125^0)\cdot\cos(-101.3099^0) - \cos(84.3726^0)\cdot\cos(-30.9637^0)}$$

$$\Rightarrow b_1 = -8$$

$$\Rightarrow \sin(\gamma_1) = \cos(\gamma_1)\, a_1 + \cos(\gamma) b_1 \quad (26)$$

$$\Rightarrow \sin(\gamma_1) = \cos(\gamma_1) - 8\cos(\gamma)$$

This is the equation of the line that passes through two points P and Q.

C. Determine the coordinates of Angular coordinate system from the Cartesian coordinate system

When we have the point $M(x, y)$in the Cartesian coordinate system and we want to transfer this point to the Angular coordinate system $M(\alpha_1, \alpha_2)$ then we follow these steps:

$$OM = \frac{R}{\cos(\alpha_1 - \alpha_2)} = \frac{R}{\cos(\alpha)} \quad (3)$$

Finding the angle (α_1)

$$x_M = OM \cdot \cos(\alpha_1) = \sqrt{(x_M)^2 + (y_M)^2} \cdot \cos(\alpha_1) \quad (11)$$

$$\Rightarrow \cos(\alpha_1) = \frac{x_M}{\sqrt{(x_M)^2 + (y_M)^2}}$$

$$\Rightarrow \alpha_1 = \pm\arccos\left(\frac{x_M}{\sqrt{(x_M)^2 + (y_M)^2}}\right) \quad (28)$$

$$y_M = OM \cdot \sin(\alpha_1) = \sqrt{(x_M)^2 + (y_M)^2} \cdot \sin(\alpha_1) \quad (10)$$

$$\Rightarrow \sin(\alpha_1) = \frac{y_M}{\sqrt{(x_M)^2 + (y_M)^2}}$$

$$\Rightarrow \alpha_1 = \pm\arcsin\left(\frac{y_M}{\sqrt{(x_M)^2 + (y_M)^2}}\right) \quad (29)$$

$$\alpha_1 = \begin{cases} \pm\arccos\left(\frac{x_M}{\sqrt{(x_M)^2 + (y_M)^2}}\right) \\ \pm\arcsin\left(\frac{y_M}{\sqrt{(x_M)^2 + (y_M)^2}}\right) \end{cases} \quad (30)$$

The equation (30) gives the position of the point M as its sign.
If $x_M \geq 0$ and $y_M \geq 0 \Rightarrow \alpha_1 \geq 0$
If $x_M \geq 0$ and $y_M < 0 \Rightarrow \alpha_1 < 0$
If $x_M < 0$ and $y_M \geq 0 \Rightarrow \alpha_1 \geq 0$
If $x_M < 0$ and $y_M < 0 \Rightarrow \alpha_1 < 0$

Finding the angle (α_2)

$$\cos(\alpha_1 - \alpha_2) = \frac{R}{OM} = \frac{R}{\sqrt{(x_M)^2 + (y_M)^2}}$$

$$\alpha_2 = \alpha_1 - \arccos\left(\frac{R}{\sqrt{(x_M)^2 + (y_M)^2}}\right) \quad (31)$$

D. Determine a vector in the Angular coordinate system in 2D space

Considering a vector in the Cartesian coordinate system

$$\overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath} \quad (32)$$

(refer to figure 9)

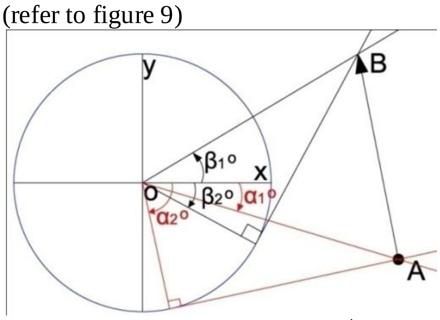


Fig.9: represents the vector $\overrightarrow{AB}$ in the Angular coordinate system.

$x_B = OB \cdot \cos(\beta_1)$ and $y_B = OB \cdot sin\,(\beta_1)$
$x_A = OA \cdot \cos(\alpha_1)$ and $y_A = OA \cdot sin\,(\alpha_1)$

$$\Rightarrow \overrightarrow{AB} = R\left(\frac{\cos(\beta_1)}{\cos(\beta)} - \frac{\cos(\alpha_1)}{\cos(\alpha)}\right)\vec{\imath} + R\left(\frac{\sin(\beta_1)}{\cos(\beta)} - \frac{\sin(\alpha_1)}{\cos(\alpha)}\right)\vec{\jmath} \quad (32')$$

Considering that
$\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$
The equation (33) is the equation of a vector in the Angular coordinate system.

Remark: if the angle α is constant and if α_1 is variable therefore the Point describes a circle of center "O".

IV. PROPERTIES OF THE ANGULAR COORDINATE SYSTEM IN 2D SPACE

In this section, we are going to determine some important geometrical and analytical properties of the Angular coordinate system in order to draw equations and curves such as $\alpha_1 = \alpha_2 + \frac{\pi}{6}$. In this paper we are taking $\alpha_1 = f(\alpha_2)$.

A. Derivation of a function

Considering a derivable curve $f(x)$ in the Cartesian coordinate system, therefore:

$$f'(x) = \lim_{x_B \to x_A} \frac{f(x_B)-f(x_A)}{x_B - x_A} \qquad (33)$$

With $x_B = x_A + \varepsilon$

$$\Rightarrow f'(x) = \lim_{x_B \to x_A} \frac{f(x_B)-f(x_A)}{x_B - x_A} = \lim_{\varepsilon \to 0} \frac{f(x_A+\varepsilon)-f(x_A)}{(x_A+\varepsilon)-x_A} = \lim_{\varepsilon \to 0} \frac{f(x_A+\varepsilon)-f(x_A)}{\varepsilon}$$

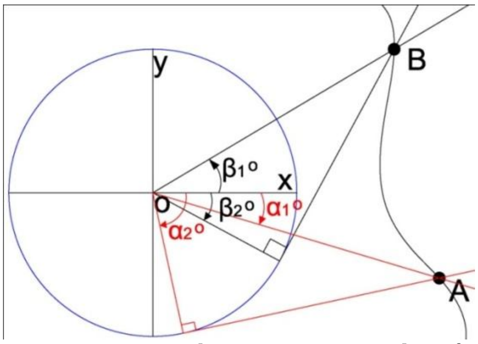


Fig.10: represents the two points A and B of a curve.

The equation (33) can be written in the Angular coordinate system as following:

$$f(x_A) = y_A = R\frac{\sin(\alpha_1)}{\cos(\alpha)}$$

$$f(x_B) = y_B = R\frac{\sin(\beta_1)}{\cos(\beta)}$$

$$x_A = R\frac{\cos(\alpha_1)}{\cos(\alpha)}$$

$$x_B = R\frac{\cos(\beta_1)}{\cos(\beta)}$$

$$\Rightarrow f'(x) = \lim_{R\frac{\cos(\beta_1)}{\cos(\beta)} \to R\frac{\cos(\alpha_1)}{\cos(\alpha)}} \left(\frac{R\frac{\sin(\beta_1)}{\cos(\beta)} - R\frac{\sin(\alpha_1)}{\cos(\alpha)}}{R\frac{\cos(\beta_1)}{\cos(\beta)} - R\frac{\cos(\alpha_1)}{\cos(\alpha)}} \right)$$

$$\Rightarrow f'(x) = \lim_{\beta \to \alpha} \left(\frac{\sin(\beta_1)\cos(\alpha)-\sin(\alpha_1)\cos(\beta)}{\cos(\beta_1)\cos(\alpha)-\cos(\alpha_1)\cos(\beta)} \right)$$

Considering that

$\beta = \alpha + \varepsilon$, $\beta_1 = \alpha_1 + \varepsilon_1$, and $\beta_2 = \alpha_2 + \varepsilon_2$

Then $\varepsilon = \varepsilon_1 - \varepsilon_2$

Therefore $\beta \to \alpha$ is equivalent to $\varepsilon \to 0$ then

$$f'(x_A) = \lim_{\varepsilon \to 0} \left(\frac{\sin(\alpha_1+\varepsilon_1)\cos(\alpha)-\sin(\alpha_1)\cos(\alpha+\varepsilon)}{\cos(\alpha_1+\varepsilon_1)\cos(\alpha)-\cos(\alpha_1)\cos(\alpha+\varepsilon)} \right) \qquad (34)$$

We have two cases:

• if $\varepsilon \to 0$ and $|\varepsilon| \geq |\varepsilon_1|$ then $\varepsilon_1 \to 0$

The equation (34) will be:

$$f'(x_A) = \lim_{\varepsilon \to 0} \left(\frac{\sin(\alpha_1)\cos(\alpha)-\sin(\alpha_1)\cos(\alpha+\varepsilon)}{\cos(\alpha_1)\cos(\alpha)-\cos(\alpha_1)\cos(\alpha+\varepsilon)} \right)$$

$$\Rightarrow f'(x_A) = \lim_{\varepsilon \to 0} \left(\frac{\sin(\alpha_1)}{\cos(\alpha_1)} \frac{\cos(\alpha)-\cos(\alpha+\varepsilon)}{\cos(\alpha)-\cos(\alpha+\varepsilon)} \right)$$

$$\Rightarrow f'(x_A) = \lim_{\varepsilon \to 0} \frac{\sin(\alpha_1)}{\cos(\alpha_1)} = \tan(\alpha_1)$$

• if $\varepsilon \to 0$ and $|\varepsilon| < |\varepsilon_1|$ then $\varepsilon_1 \to 0^{\pm}$

The equation (34) will be:

$$f'(x_A) = \lim_{\varepsilon_1 \to 0^{\pm}} \left(\frac{\sin(\alpha_1+\varepsilon_1)\cos(\alpha)-\sin(\alpha_1)\cos(\alpha)}{\cos(\alpha_1+\varepsilon_1)\cos(\alpha)-\cos(\alpha_1)\cos(\alpha)} \right)$$

$$\Rightarrow f'(x_A) = \lim_{\varepsilon_1 \to 0^{\pm}} \left(\frac{\sin(\alpha_1+\varepsilon_1)-\sin(\alpha_1)}{\cos(\alpha_1+\varepsilon_1)-\cos(\alpha_1)} \right)$$

$$\Rightarrow f'(x_A) = \lim_{\varepsilon_1 \to 0^{\pm}} \left(\frac{\sin(\alpha_1)\cos(\varepsilon_1)+\cos(\alpha_1)\sin(\varepsilon_1)-\sin(\alpha_1)}{\cos(\alpha_1)\cos(\varepsilon_1)-\sin(\alpha_1)\sin(\varepsilon_1)-\cos(\alpha_1)} \right) \qquad (35)$$

We have

$$\cos(\varepsilon_1) = 1 - \frac{\varepsilon_1^{\,2}}{2} \qquad (36)$$

$$\sin(\varepsilon_1) = \varepsilon_1 - \frac{\varepsilon_1^{\,3}}{6} = \varepsilon_1 \qquad (37)$$

By putting the equations (36) and (37) in the equation (35) we obtain finally:

$$f'(x_A) = -\frac{1}{\tan(\alpha_1)} = -cotan(\alpha_1)$$

B. Study of ε in different regions in the space

The space of the Angular coordinate system can be divided into 8 regions as shown in figure 11.

The study of regions helps us to understand how a curve can be described in the space and how it is moving.

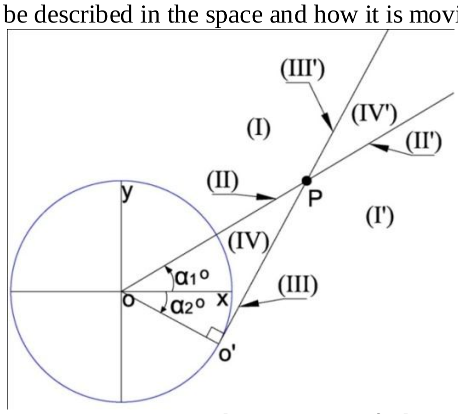


Fig.11: represents the regions of the Angular coordinate system space.

These regions help us to study the variation of the angle ε, ε_1 and ε_2 in the space.

Remark:

$\beta = \alpha + \varepsilon$

$\beta_1 = \alpha_1 + \varepsilon_1$; $\beta_2 = \alpha_2 + \varepsilon_2$; $\varepsilon = \varepsilon_1 - \varepsilon_2$

If $\beta \to \alpha$ then $\to 0$, which means that the Point B is going to the Point A as shown in the figure 10.

•Region (I):

As shown in the figure 12, the point A is in the region (I) (at the left part of (OBC) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 < 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 < 0$

$$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 \Rightarrow \begin{cases} \varepsilon < 0 \; if \; |\varepsilon_1| > |\varepsilon_2| \\ \varepsilon \geq 0 \; if \; |\varepsilon_1| \leq |\varepsilon_2| \end{cases}$$

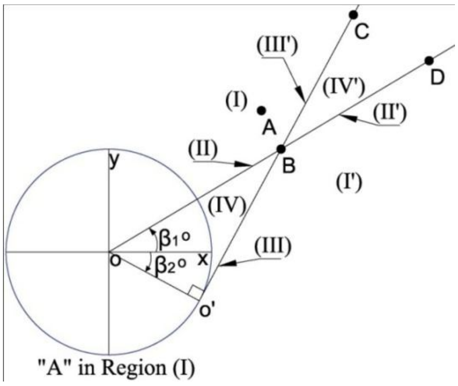


Fig.12: represents the point A in the region (I) of the Angular coordinate system space.

•Region (I'):

As shown in the figure 13, the point A is in the region (I') (at the right part of (O'BD) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 > 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 > 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 \Rightarrow \begin{cases} \varepsilon < 0 \; if \; |\varepsilon_1| < |\varepsilon_2| \\ \varepsilon \geq 0 \; if \; |\varepsilon_1| \geq |\varepsilon_2| \end{cases}$

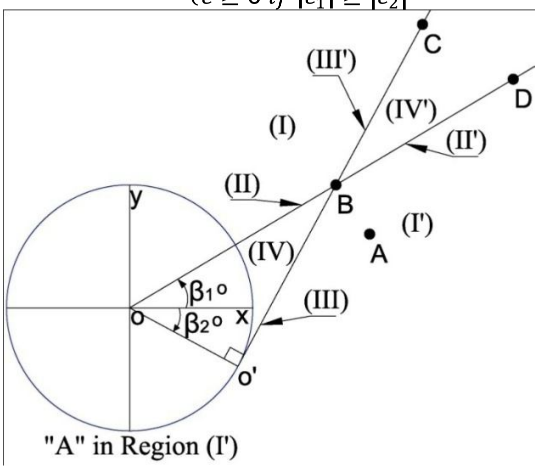


Fig.13: represents the point A in the region (I') of the Angular coordinate system space.

•Region (IV):

As shown in the figure 14, the point A is in the region (IV) (in the triangle (OBO') outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 > 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 < 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 > 0 \Rightarrow |\varepsilon| > |\varepsilon_1|$

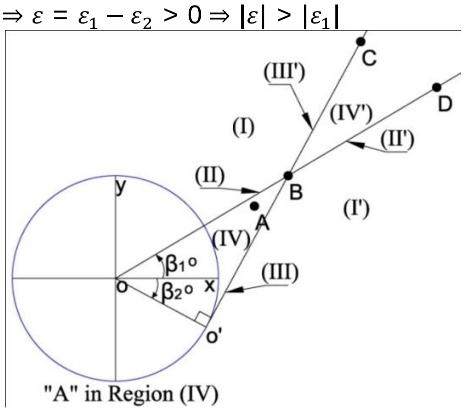


Fig.14: represents the point A in the region (IV) of the Angular coordinate system space.

•Region (IV'):

As shown in the figure 15, the point A is in the region (IV') (in the region (CBD) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 < 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 > 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 < 0 \Rightarrow |\varepsilon| > |\varepsilon_1|$

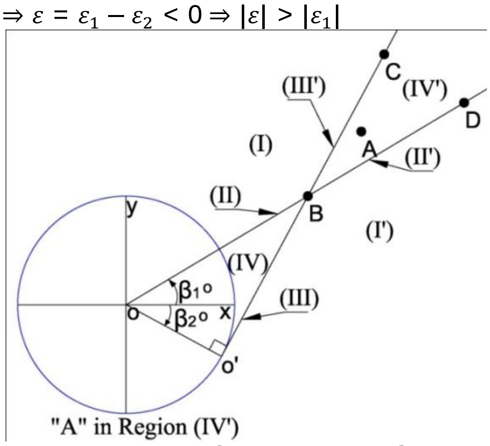


Fig.15: represents the point A in the region (IV') of the Angular coordinate system space.

•Region (II):

As shown in the figure 16, the point A is in the region (II) (in the segment (OB) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 = 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 < 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 > 0 \Rightarrow |\varepsilon| > |\varepsilon_1|$

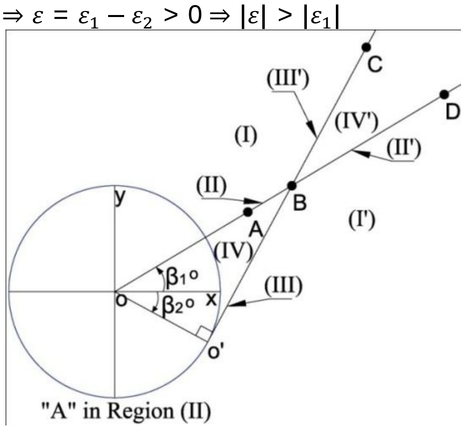


Fig.16: represents the point A in the region (II) of the Angular coordinate system space.

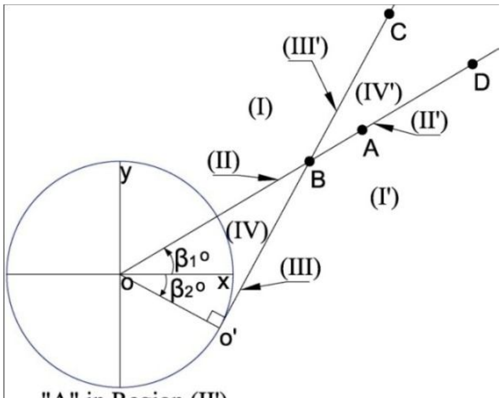
•Region (II'):

As shown in the figure 17, the point A is in the region (II') (in the segment (BD) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 = 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 > 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 < 0 \Rightarrow |\varepsilon| > |\varepsilon_1|$



"A" in Region (II')

Fig.17: represents the point A in the region (II') of the Angular coordinate system space.

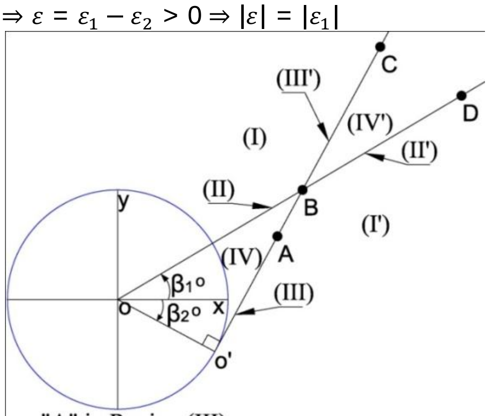
•Region (III):

As shown in the figure 18, the point A is in the region (III) (in the segment (O'B) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 > 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 = 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 > 0 \Rightarrow |\varepsilon| = |\varepsilon_1|$



"A" in Region (III)

Fig.18: represents the point A in the region (III) of the Angular coordinate system space.

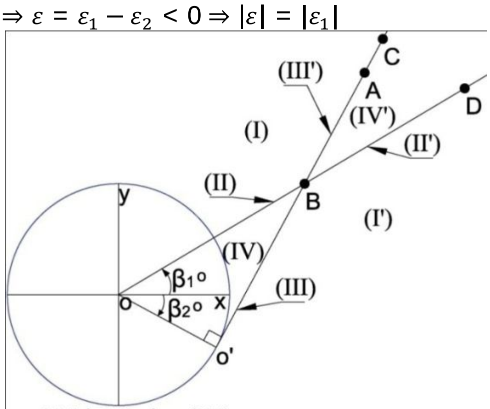
•Region (III'):

As shown in the figure 19, the point A is in the region (III') (in the segment (BC) outside the circle)

Therefore:

$\varepsilon_1 = \beta_1 - \alpha_1 < 0$ and $\varepsilon_2 = \beta_2 - \alpha_2 = 0$

$\Rightarrow \varepsilon = \varepsilon_1 - \varepsilon_2 < 0 \Rightarrow |\varepsilon| = |\varepsilon_1|$



"A" in Region (III')

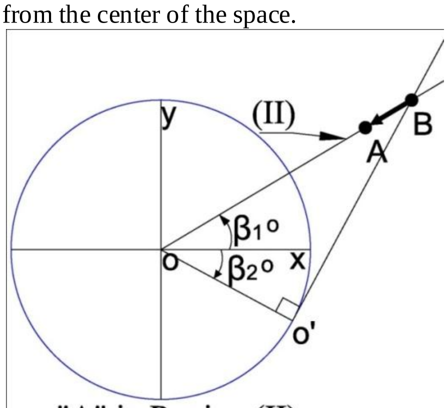
Fig.19: represents the point A in the region (III') of the Angular coordinate system space.

ε has a varied sign and magnitude according to the movement of the curve in the Angular coordinate system.

C. Different movement of the curve

C.1. Approach of the curve to the center of the angular coordinate system.

If the angle β_1 is constant and the angle β_2 is approaching from the angle β_1, then we say that the curve is approaching from the center of the space.

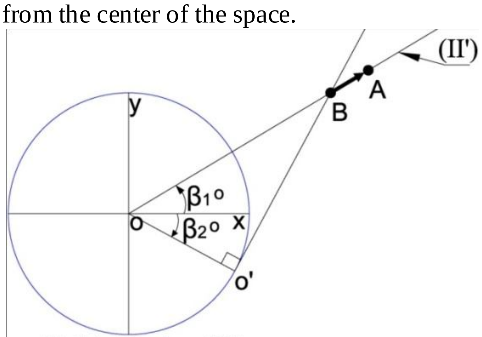


"A" in Region (II)

Fig.20: represents the point A in the region (II) of the Angular coordinate system space.

C.2. Separate (recede) of the curve from the center of the angular coordinate system.

If the angle β_1 is constant and the angle β_2 is separating from the angle β_1, then we say that the curve is separating from the center of the space.



"A" in Region (II')

Fig.21: represents the point A in the region (II') of the Angular coordinate system space.

C.3. Approach of the curve to the tangent of the unit circle of the angular coordinate system.

If the angle β_2 is constant and the angle β_1 is approaching from the angle β_2, then we say that the curve is approaching to the tangent of the unit circle of the angular coordinate system.

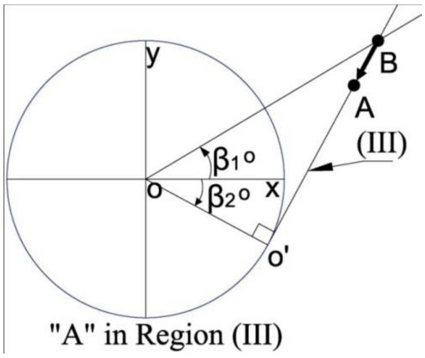


Fig.22: represents the point A in the region (III) of the Angular coordinate system space.

C.4. Separate (recede) of the curve from the tangent of the unit circle of the angular coordinate system.

If the angle β_2 is constant and the angle β_1 is separating from the angle β_2, then we say that the curve is separating from the tangent of the unit circle of the angular coordinate system.

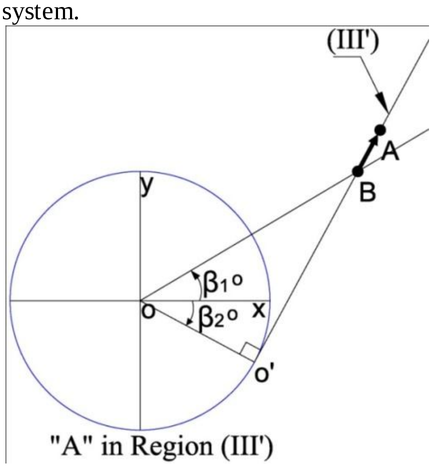


Fig.23: represents the point A in the region (III') of the Angular coordinate system space.

C.5. Clockwise turn of the curve from the unit circle of the angular coordinate system.

We define

• $d\beta_2 = \beta''_2 - \beta'_2$ (38)

• $d\beta_1 = \beta''_1 - \beta'_1$ (39)

• $\frac{d\beta_1}{d\beta_2} = \frac{\beta''_1 - \beta'_1}{\beta''_2 - \beta'_2}$ (40)

The equation (40) is the variation of β_1 regarding β_2 with always the verified condition:

$0 \le \beta_1 - \beta_2 < 90^0$ (41)

If $d\beta_1 < 0$, then the curve is turning Clockwise.
If $d\beta_2 < 0$, then the curve is turning Clockwise.

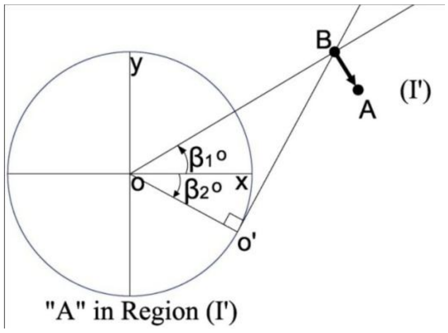


Fig.24: represents the point A in the region (I') of the Angular coordinate system space.

C.6. Counterclockwise turn of the curve from the unit circle of the angular coordinate system.

We define

• $d\beta_2 = \beta''_2 - \beta'_2$ (38)

• $d\beta_1 = \beta''_1 - \beta'_1$ (39)

• $\frac{d\beta_1}{d\beta_2} = \frac{\beta''_1 - \beta'_1}{\beta''_2 - \beta'_2}$ (40)

The equation (40) is the variation of β_1 regarding β_2 with always the verified condition:

$0 \le \beta_1 - \beta_2 < 90^0$ (41)

If $d\beta_1 > 0$, then the curve is turning Counterclockwise.
If $d\beta_2 > 0$, then the curve is turning Counterclockwise.

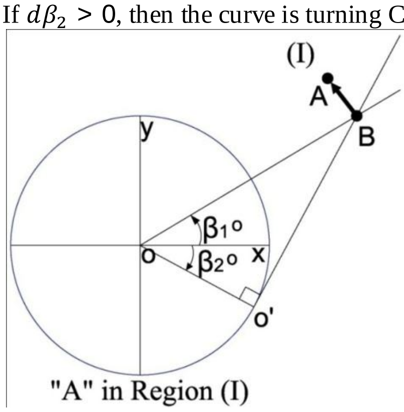


Fig.25: represents the point A in the region (I) of the Angular coordinate system space.

C.7. Approach of the curve to the unit circle of the angular coordinate system.

If the angle β_1 and β_2 are approaching to each other, then we say that the curve is approaching to the unit circle of the angular coordinate system.

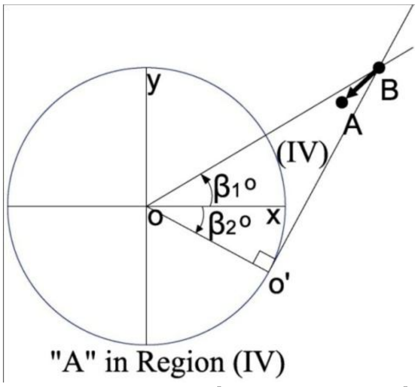


Fig.26: represents the point A in the region (IV) of the Angular coordinate system space.

C.8. Separate (recede) of the curve to the unit circle of the angular coordinate system.

If the angle β_1 and β_2 are separating for each other, then we say that the curve is separating from the unit circle of the angular coordinate system.

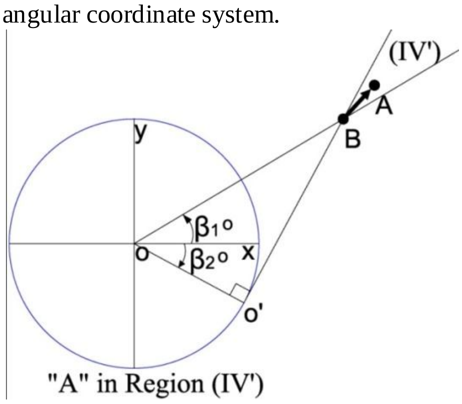


Fig.27: represents the point A in the region (IV') of the Angular coordinate system space.

D. Study of the derivation of curves in the angular coordinate system

As we have 8 regions (refer to figure 26) we consider a curve in the Angular coordinate system and a point "P" belongs to the curve. The nearest point to the point "P" must be in one of the possible 8 regions of the space

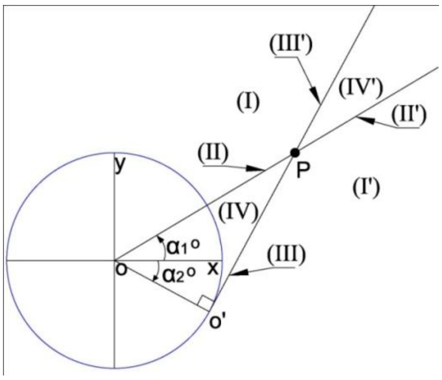


Fig.28: represents the regions of the Angular coordinate system space.

• $region\ I \begin{cases} d\alpha_1 > 0 \\ d\alpha_2 > 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} > 0$ (42)

The curve is turning in the counterclockwise direction.

• $region\ I' \begin{cases} d\alpha_1 < 0 \\ d\alpha_2 < 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} > 0$ (43)

The curve is turning in the clockwise direction.

• $region\ II \begin{cases} d\alpha_1 = 0 \\ d\alpha_2 > 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} = 0$ (44)

The curve is approaching to the center.

• $region\ II' \begin{cases} d\alpha_1 = 0 \\ d\alpha_2 < 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} = 0$ (45)

The curve is separating (receding) from the center.

• $region\ III \begin{cases} d\alpha_1 < 0 \\ d\alpha_2 = 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} = -\infty$ (46)

The curve is approaching to the tangent of the unit circle.

• $region\ III' \begin{cases} d\alpha_1 > 0 \\ d\alpha_2 = 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} = +\infty$ (47)

The curve is separating (receding) from the tangent of the unit circle.

• $region\ IV \begin{cases} d\alpha_1 < 0 \\ d\alpha_2 > 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} < 0$ (48)

The curve is approaching to the unit circle.

• $region\ IV' \begin{cases} d\alpha_1 > 0 \\ d\alpha_2 < 0 \end{cases} \Rightarrow \frac{d\alpha_1}{d\alpha_2} < 0$ (49)

The curve is separating from the unit circle.

• If the curve is approaching, then $\alpha = \alpha_1 - \alpha_2$ is decreasing.
• If the curve is separating (receding), then $\alpha = \alpha_1 - \alpha_2$ is increasing.
• If $d\alpha_1$ change the sign, then the curve change the turning. The same for $d\alpha_2$.

E. Study of the summit of a curve in the angular coordinate system

We define a summit when the $d\alpha_1$ and $d\alpha_2$ change their signs (for example $d\alpha_1 > 0 \rightarrow d\alpha_1 < 0$)

We have many cases to study as following:

E.1. Minimum summit

When both $d\alpha_1$ and $d\alpha_2$ change their signs

$$\begin{cases} d\alpha_1 < 0 \rightarrow d\alpha_1 > 0 \\ d\alpha_2 < 0 \rightarrow d\alpha_2 > 0 \end{cases} \quad (50)$$

In the case of equation (50) we say that the curve has minimum summit as in figure 29.

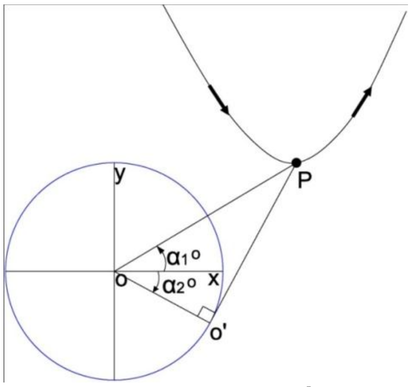


Fig.29: represents a curve with a minimum summit.

E.2. Maximum summit

When both $d\alpha_1$ and $d\alpha_2$ change their signs

$$\begin{cases} d\alpha_1 > 0 \rightarrow d\alpha_1 < 0 \\ d\alpha_2 > 0 \rightarrow d\alpha_2 < 0 \end{cases} \quad (51)$$

In the case of equation (51) we say that the curve has maximum summit as in figure 30.

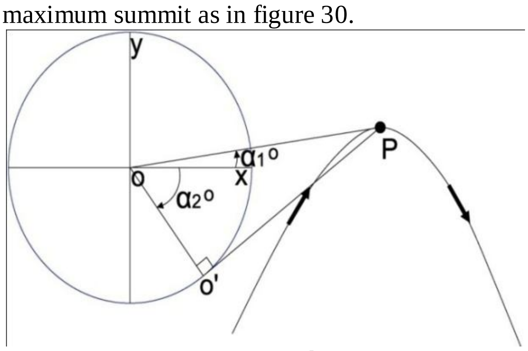


Fig.30: represents a curve with a minimum summit.

E.3. Point of changing sign

When one of two angles $d\alpha_1$ or $d\alpha_2$ changes its sign

$$\bullet \begin{cases} d\alpha_1 > 0 \rightarrow d\alpha_1 > 0 \\ d\alpha_2 > 0 \rightarrow d\alpha_2 < 0 \end{cases} \quad (52)$$

In the case of equation (52) we say that the curve has changed the direction from region (I) to region (IV') (refer to figure 28).

$$\bullet \begin{cases} d\alpha_1 > 0 \rightarrow d\alpha_1 < 0 \\ d\alpha_2 > 0 \rightarrow d\alpha_2 > 0 \end{cases} \quad (53)$$

In the case of equation (53) we say that the curve has changed the direction from region (I) to region (IV) (refer to figure 28).

$$\bullet \begin{cases} d\alpha_1 > 0 \rightarrow d\alpha_1 = 0 \\ d\alpha_2 > 0 \rightarrow d\alpha_2 > 0 \end{cases} \quad (54)$$

In the case of equation (54) we say that the curve has changed the direction from region (I) to region (IV) (refer to figure 28).

$$\bullet \begin{cases} d\alpha_1 > 0 \rightarrow d\alpha_1 > 0 \\ d\alpha_2 > 0 \rightarrow d\alpha_2 = 0 \end{cases} \quad (55)$$

In the case of equation (55) we say that the curve has changed the direction from region (I) to region (III') (refer to figure 28).

The same principal can be applied for other signs.

V. STEPS OF THE STUDY FOR A FUNCTION IN THE ANGULAR COORDINATE SYSTEM

We have mainly 8 principal steps:

1- Find the Domain of definition of the curve using the following equation $0 \leq \alpha_1 - \alpha_2 < 90^0$. Try to reduce the domain as possible.
2- Study of the movement and estrangement of the curve with $\alpha = \alpha_1 - \alpha_2$.
3- Study of the derivation of the curve $d\alpha_1/d\alpha_2$, $d\alpha_1$ and $d\alpha_2$.
4- Study the variation of $d\alpha_1$ and $d\alpha_2$ in order to know the regions of the curve.
5- Study the asymptote if it exists in the domain.
6- Draw the table of definition
7- Find if there are particular points
8- Draw the graph

VI. EXAMPLES OF APPLICATIONS

In this section, some examples are done in order to give an idea about how to study a curve in the Angular coordinate system similar to the Cartesian coordinate system.

A. Example 1: $\alpha_1 = \alpha_2 + 30^0$

1- Find the Domain of definition of the curve using the following equation $0 \leq \alpha_1 - \alpha_2 < 90^0$. Try to reduce the domain as possible.

• $0 \leq \alpha_1 - \alpha_2 < 90^0 \Rightarrow 0 \leq \alpha_2 + 30^0 - \alpha_2 < 90^0$
$\Rightarrow 0 \leq \alpha_2 + 30^0 - \alpha_2 < 90^0 \Rightarrow 0 \leq 30^0 < 90^0$

The domain of definition is always correct whatever is the angles
$\Rightarrow \alpha_1 =]-\infty; +\infty[$

Or the equation is periodic with
$\alpha_1(\alpha_2 + 360^0) = \alpha_2 + 30^0$

Therefore the period is equal to $2\pi \Rightarrow \alpha_1 = [0; 2\pi]$ or $\alpha_1 = [0^0; 360^0]$

2- Study of the movement and estrangement of the curve with $\alpha = \alpha_1 - \alpha_2$.

• $\alpha = \alpha_1 - \alpha_2 = 30^0$ so it is constant therefore we can deduce that it describe a circle with center "O".

3- Study of the derivation of the curve $d\alpha_1/d\alpha_2$, $d\alpha_1$ and $d\alpha_2$.

• $\frac{d\alpha_1}{d\alpha_2} = \frac{d(\alpha_2+30^0)}{d\alpha_2} = 1 > 0$

Or $\alpha_2 = [0^0; 360^0]$
$d\alpha_1 = \alpha''_1 - \alpha'_1 > 0$

Therefore we can say that the curve is turning into the counterclockwise direction ↶. Therefore there is no summit.

4- Study the variation of $d\alpha_1$ and $d\alpha_2$ in order to know the regions of the curve.

• $d\alpha_1 = \alpha''_1 - \alpha'_1 > 0$
$d\alpha_2 = \alpha''_2 - \alpha'_2 > 0$

The curve is in the region (I)

5- Study the asymptote if it exists in the domain.

• There is no asymptote because
$\lim_{\alpha_2 \to 0} \alpha_1 = 30^0$
$\lim_{\alpha_2 \to 360} \alpha_1 = 390^0 = 30^0$

6- Draw the table of definition

α_2	$0°$	$360°$
α	constant	
$\frac{d\alpha_1}{d\alpha_2}$	+	
$d\alpha_1$	+	
$d\alpha_2$	+	
α_1	$30°$ ↗	$390°$

7- Find if there are particular points

α_2	0^0	90^0	180^0	270^0	360^0
α_1	30^0	120^0	210^0	300^0	390^0

8- Draw the graph

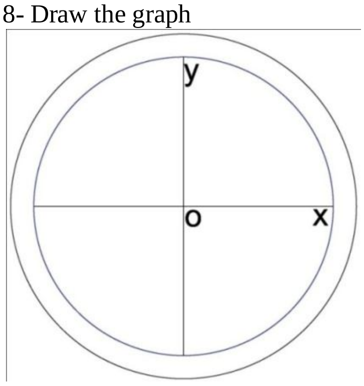


Fig.31: represents the curve $\alpha_1 = \alpha_2 + 30^0$.

9- Function programmed in Matlab

```
%----Beginning of the Program----
%Angular Coordinate System
%Principal Circle
r=1; %Radius of the central circle
c2=0:0.1:360; % in radiant from 0 to 2*PI
c1=c2;
xc=r.*cos(pi*c1/180)./cos(pi*c1/180-pi*c2/180);
yc=r.*sin(pi*c1/180)./cos(pi*c1/180-pi*c2/180);

%Function A1=F(A2)
A2=-180:0.1:180; %Domain of definition of "A2"
A1=A2+30;%Equation
%Drawing in the Cartesian Coordinate system which is equivalent to the Angular Coordinate system in this Program
x=r.*cos(pi*A1/180)./cos(pi*A1/180-pi*A2/180);
y=r.*sin(pi*A1/180)./cos(pi*A1/180-pi*A2/180);

%Plot the result
figure(1);
plot(xc,yc,'r');hold on;
plot(x,y);grid;
axis([-10 10 -10 10])
%----End of the Program----
```

B. Example 2: $\alpha_1 = 1/\alpha_2$

1- Find the Domain of definition of the curve using the following equation $0 \le \alpha_1 - \alpha_2 < 90^0$. Try to reduce the domain as possible.

• The first condition is that $\alpha_2 \neq 0$

$0 \le \alpha_1 - \alpha_2 < 90^0 \Rightarrow 0 \le \frac{1}{\alpha_2} - \alpha_2 < 90^0$

$\Rightarrow 0 \le \frac{1-\alpha_2{}^2}{\alpha_2} < 90^0 \Rightarrow 0 \le 1 - \alpha_2{}^2 < 90^0\alpha_2$

$0 \le 1 - \alpha_2{}^2 \Rightarrow -1 \le \alpha_2 \le 1$

And $1 - \alpha_2{}^2 < 90^0\alpha_2 \Rightarrow \alpha_2{}^2 + 90^0\alpha_2 - 1 > 0$

$\Rightarrow (\alpha_2 + 45^0 - \sqrt{2026}) \cdot (\alpha_2 + 45^0 + \sqrt{2026}) > 0$

α_2	$-\infty$	$-45-\sqrt{2026}$	$-45+\sqrt{2026}$	$+\infty$
$\alpha_2+45-\sqrt{2026}$	-	-	+	
$\alpha_2+45+\sqrt{2026}$	-	+	+	
	+	-	+	

Therefore the domain of definition is limited by:

$\alpha_2 \in]-45+\sqrt{2026};1]$

2- Study of the movement and estrangement of the curve with $\alpha = \alpha_1 - \alpha_2$.

• $\alpha = \frac{1-\alpha_2{}^2}{\alpha_2} \ge 0$

$\frac{d\alpha}{d\alpha_2} = \frac{-1-\alpha_2{}^2}{\alpha_2{}^2} < 0$ Therefore we can say that the curve is approaching from the unit circle

3- Study of the derivation of the curve $d\alpha_1/d\alpha_2$, $d\alpha_1$ and $d\alpha_2$.

•$\frac{d\alpha_1}{d\alpha_2} = \frac{-1}{\alpha_2{}^2} < 0$ Then we have two possible regions (IV) and (IV').

4- Study the variation of $d\alpha_1$ and $d\alpha_2$ in order to know the regions of the curve.

• $d\alpha_2 = 1 > 0 \Rightarrow d\alpha_1 < 0$ Then the region is (IV) and we can say that the curve is approaching from the unit circle

5- Study the asymptote if it exists in the domain.

• $\lim_{\alpha_2 \to 1} \alpha_1 = 1^0$
$\lim_{\alpha_2 \to -45+\sqrt{2026}} \alpha_1 = 90.0111^0$
$\lim_{\alpha_2 \to -45+\sqrt{2026}} \alpha = 90^0$
There is an asymptote.

6- Draw the table of definition

α_2	$-45+\sqrt{2026}$	1^o
α'	-	
α	↘	
$\frac{d\alpha_1}{d\alpha_2}$	-	
$d\alpha_1$	-	
$d\alpha_2$	+	
α_1	$\frac{1}{-45+\sqrt{2026}}$ ↘	1^o

7- Find if there are particular points

α_2	0.02^0	0.04^0	0.05^0	0.1^0	1^0
α_1	50^0	25^0	20^0	10^0	1^0

8- Draw the graph

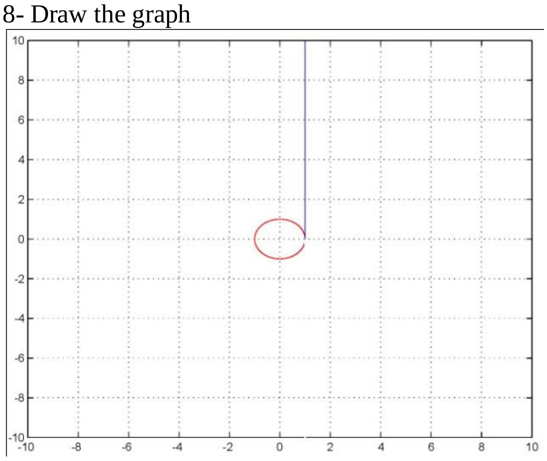


Fig.32: represents the curve $\alpha_1 = 1/\alpha_2$

9- Function programmed in Matlab

```
%----Beginning of the Program----
%Angular Coordinate System
%Principal Circle
r=1; %Radius of the central circle
c2=0:0.1:360; % in radiant from 0 to 2*PI
c1=c2;
xc=r.*cos(pi*c1/180)./cos(pi*c1/180-
pi*c2/180);
yc=r.*sin(pi*c1/180)./cos(pi*c1/180-
pi*c2/180);

%Function A1=F(A2)
%A1=1/A2
A2=0.011109:0.00001:1;      %Domain      of
definition of "A2"
A1=1./A2;%Equation
%Drawing  in  the  Cartesian  Coordinate
system which is equivalent to the Angular
Coordinate system in this Program
x=r.*cos(pi*A1/180)./cos(pi*A1/180-
pi*A2/180);
y=r.*sin(pi*A1/180)./cos(pi*A1/180-
pi*A2/180);

%Plot the result
figure(1);
plot(xc,yc,'r');hold on;
plot(x,y);grid;
axis([-10 10 -10 10])
%----End of the Program----
```

C. Example 3: $\alpha_1 = \alpha_2{}^2 - 10$

1- Find the Domain of definition of the curve using the following equation $0 \le \alpha_1 - \alpha_2 < 90^0$. Try to reduce the domain as possible.

•$0 \le \alpha_2{}^2 - 10 - \alpha_2 < 90^0$

$$\Rightarrow 0 \le \left(\alpha_2 - \frac{\sqrt{41}+1}{2}\right)\left(\alpha_2 - \frac{-\sqrt{41}+1}{2}\right) < 90^0$$

We have two cases:

α_2	$-\infty$	$\frac{+1-\sqrt{41}}{2}$		$\frac{1+\sqrt{41}}{2}$	$+\infty$
$\alpha_2-\frac{1+\sqrt{41}}{2}$	-		-	0	+
$\alpha_2-\frac{+1-\sqrt{41}}{2}$	-	0	+		+
	+		-		+

a-

The first domain is $]-\infty;\frac{-\sqrt{41}+1}{2}]\cup[\frac{\sqrt{41}+1}{2};+\infty[$

b- $\alpha_2{}^2 - 10 - \alpha_2 < 90^0$

$$\Rightarrow \left(\alpha_2 - \frac{1}{2} - \frac{\sqrt{401}}{2}\right)\left(\alpha_2 - \frac{1}{2} + \frac{\sqrt{401}}{2}\right) < 0$$

α_2	$-\infty$	$\frac{+1-\sqrt{401}}{2}$		$\frac{1+\sqrt{401}}{2}$	$+\infty$
$\alpha_2-\frac{1+\sqrt{41}}{2}$	-		-	0	+
$\alpha_2-\frac{+1-\sqrt{41}}{2}$	-	0	+		+
	+		-		+

The second domain is $]-\frac{\sqrt{401}-1}{2};\frac{\sqrt{401}+1}{2}[$

Therefore the common domain of α_2 is

$]-\frac{\sqrt{401}-1}{2};\frac{-\sqrt{41}+1}{2}]\cup[\frac{\sqrt{41}+1}{2};\frac{\sqrt{401}+1}{2}[$

$=]-9.512;-2.7015]\cup[3.7015;10.512[$

And the domain of $\alpha_1$is

$=]-80.487;-2.7015]\cup[3.7015;100.51249[$

2- Study of the movement and estrangement of the curve with $\alpha = \alpha_1 - \alpha_2$.

• $\alpha = \alpha_1 - \alpha_2 = \alpha_2{}^2 - 10 - \alpha_2$

$$\frac{d\alpha}{d\alpha_2} = 2\alpha_2 - 1$$

For $\alpha_2 > \frac{1}{2}$ the curve is separating

For $\alpha_2 < \frac{1}{2}$ the curve is approaching

3- Study of the derivation of the curve $d\alpha_1/d\alpha_2$, $d\alpha_1$ and $d\alpha_2$.

•$\frac{d\alpha_1}{d\alpha_2} = 2\alpha_2$

If $\alpha_2 > 0 \Rightarrow \frac{d\alpha_1}{d\alpha_2} > 0$

If $\alpha_2 < 0 \Rightarrow \frac{d\alpha_1}{d\alpha_2} < 0$

4- Study the variation of $d\alpha_1$ and $d\alpha_2$ in order to know the regions of the curve.

• $\frac{d\alpha_1}{d\alpha_2} = 2\alpha_2$

If $\alpha_2 > 0 \Rightarrow \frac{d\alpha_1}{d\alpha_2} > 0$ then the possible regions are (I) and (I').
Therefore $d\alpha_1 > 0$ the only possible region is (I)
If $\alpha_2 < 0 \Rightarrow \frac{d\alpha_1}{d\alpha_2} < 0$ then the possible regions are (IV) and (IV'). So $d\alpha_1 < 0$ therefore the possible region is (IV).

5- Study the asymptote if it exists in the domain.

• $\lim_{\alpha_2 \to -\frac{\sqrt{401}-1}{2}} \alpha = 90^0$ Therefore it is an asymptote

$\lim_{\alpha_2 \to \frac{-\sqrt{41}+1}{2}} \alpha = 0^0$

$\lim_{\alpha_2 \to \frac{\sqrt{401}+1}{2}} \alpha = 90^0$ Therefore it is an asymptote

6- Draw the table of definition

α_2	$\frac{+1-\sqrt{401}}{2}$		$\frac{+1-\sqrt{41}}{2}$	$\frac{+1}{2}$	$\frac{1+\sqrt{41}}{2}$		$\frac{1+\sqrt{401}}{2}$
α'		-		0		+	
α		↘	0		0	↗	
$\frac{d\alpha_1}{d\alpha_2}$		+				+	
$d\alpha_1$		-				+	
$d\alpha_2$		-				+	
α_1	80.48	↘	-2.7		3.7	↗	100.5

7- Find if there are particular points

α_2	-9^0	-5^0	4^0	7^0	50^0
α_1	71^0	15^0	6^0	39^0	2490^0

8- Draw the graph

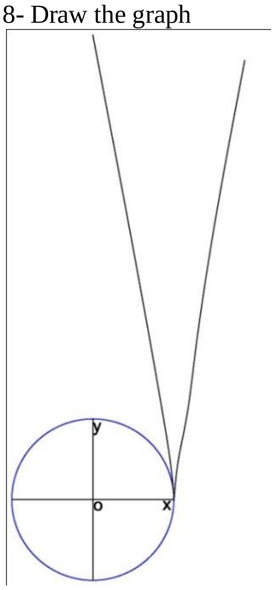


Fig.33: represents the curve $\alpha_1 = {\alpha_2}^2 - 10$

D. Example 4: $\alpha_1 = \alpha_2 + 80^0 + cos(\alpha_2)$
The shape of this function will be as following:

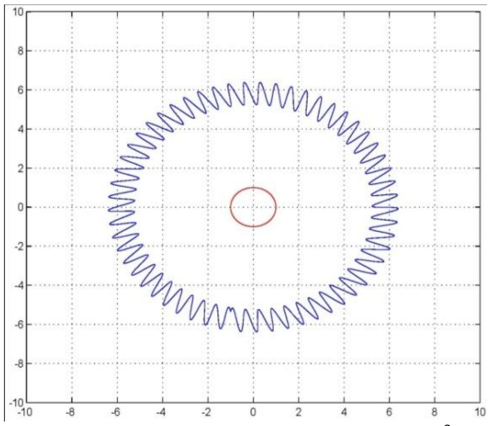


Fig.34: represents the curve $\alpha_1 = \alpha_2 + 80^0 + \cos(\alpha_2)$

E. Example 5: $\alpha_1 = \alpha_2 + 80^0 + 4 \cdot cos(\alpha_2)$
The shape of this function will be as following:

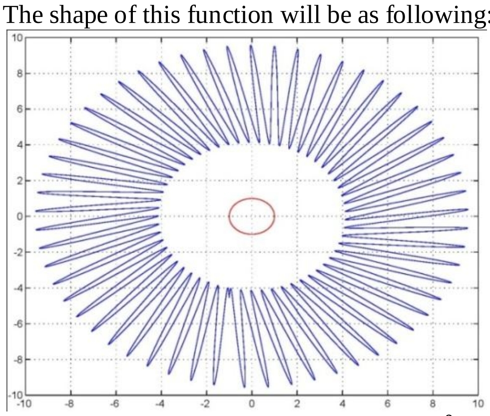


Fig.35: represents the curve $\alpha_1 = \alpha_2 + 80^0 + 4\cos(\alpha_2)$

VII. CONCLUSION

The Angular coordinate system is an original study introduced by the author in the mathematical domain.
The main goal of introducing this system is to facilitate the study of some curves that are not easy to study in the other coordinate systems. Moreover, the author proposed an instrument that gives the position of any point in the 2D space. By using this coordinate system, we can describe the whole universe in 2D space using two angles only formed by the center of a unit circle of the universe. A deep study is introduced in order to know how the angular coordinate system works and how to study curves in this system. Some examples are introduced in order to give an idea about how to study a curve in all details. The Angular coordinate system can have many applications in mathematics, physics and engineering and it can facilitate the study of many complicated curves that are almost difficult to study in the traditional coordinate system.
And finally as a conclusion, this coordinate system is very important in physics in which we can consider a black hole as the unit circle and we have to study the behavior of the external universe according to this black hole. In this way we can form a universe in which the black hole is the center of it and we form curves of defined objects such as rocks, planets, stars and so on.

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