On the group classification and conservation laws of the self-adjoint of a family Benjamin-Bona-Mahony equations

M. S. Bruzón and M.L. Gandarias

***Abstract*—In this paper, we consider a family of Benjamin-Bona-Mahony equation with strong nonlinear dispersive which is of considerable interest in mathematical physics. We determine the subclass of equations which are self-adjoint. By using a general theorem on conservation laws proved in [21], [22] conservation laws for this equation are presented.**

***Index Terms*—Symmetries, Partial differential equation, self-adjointness, conservation laws**

I. Introduction

The partial differential equations govern many phenomena that happen in the nature and they are indispensable for the advance of the engineering and technology. Essentially all the fundamental equations are nonlinear and, in general, such nonlinear equations are often very difficult to solve explicitly. Symmetry group techniques provide methods for obtaining solutions of these equations. These methods have several applications, for example, in the study of nonlinear partial differential equations that admit conservation laws which arise in many disciplines of the applied sciences.

Benjamin *et al* [3] proposed the regularised long wave (RLW) equation, or Benjamin-Bona-Mahony equation (BBM),

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

as an alternative model to the Korteweg–de Vries equation for the long wave motion in nonlinear dispersive systems. These authors argued that both equations are valid at the same level of approximation, but that BBM equation does have some advantages over the KdV equation from the computational mathematics viewpoint.

In order to understand the role of nonlinear dispersion in the formation of patterns in an undular bore, Yalong [31] introduced and studied a family of BBM-like equations with nonlinear dispersion, B(m, n) equations

$$u_t + (u^m)_x - (u^n)_{xxt} = 0, \qquad m, n > 1.$$

In [31], the exact solitary-wave solutions with compact support and exact special solutions with solitary patterns of the equations were derived.

In [26] introduced the family of BBM equation with strong nonlinear dispersive B(m,n) equation:

$$u_t + u_x + a\,(u^m)_x + (u^n)_{xxt} = 0,$$

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by using an algebraic method the authors obtained solitary pattern solutions. The case $n = 1$ and $m = 2$ corresponds to the BBM equation, [3]. This equation is an alternative to the Kortewegde Vries (KdV) equation and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. The BBM equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves and therefore it has some advantages compared with the KdV equation.

Clarkson [13] showed that the similarity reduction of the equation (1) for $m = 3$, $n = 1$ and $a = \frac{1}{3}$, obtained by using the classical Lie group method reduces the partial differential equation (PDE) to an ordinary differential equation (ODE) of Painlevé type; whereas the PDE doesn't possess the Painlevé property for PDEs as defined by Weiss *et al* [27]. The author proved that the only non-constant similarity reductions of this equation obtainable either using the classical Lie method or the direct method, due to Clarkson and Kruskal [14], are the travelling wave solutions.

Conservation laws are fundamental laws of physics that maintain that a certain quantity will not change in time during physical processes. Nonlinear partial differential equations that admit conservation laws which arise in many disciplines of the applied sciences.

The investigation of conservation laws of the Korteveg-de Vries equation was the starting point of the discovery of a number of techniques to solve evolutionary equations (Miura transformation, Lax pair, inverse scattering technique, bi-Hamiltonian structures). The existence of a large number of conservation laws of a PDE is a strong indication of its integrability [33].

The knowledge of conservation laws is useful in the numerical integration of PDEs, for example, to control numerical errors. Numerical experiments first carried out by Abdulloev et a1 (1976) and then by others (see Bona et a1 1983) show that the BBM equation admits soliton solutions whose interaction is inelastic though close to elastic. Considering the BBM equation as a "deformation" of the Kdv equation, we see that the latter displays surprising stability of its seemingly fragile mathematical properties. Therefore a natural question arises as to whether the behaviour of the solutions of the BBM equation can be explained in terms of conservation laws.

Below, if the BBM equation is written as $u_{xxt} = u_t - uu_x$ (it takes this form after replacing u by $-1 - u$ in the original version). Olver (1979) showed that this equation has no other conserved densities depending only on $x, u, u_x, u_{xx}, \ldots$ than those indicated by Benjamin et a1 (1972): u (mass), $(u^2 + u_x^2)/2$ (energy), and $u^3/3$ (momentum). This result, however, does not imply that the BBM equation is not a completely integrable Hamiltonian system, since there might exist other conserved densities which depend also on t and t-derivatives of u and u_x. Note that from the point of view adopted by Olver these conserved densities, if they exist, can be considered also as functions of x, u and$x$-derivatives of u which are however non-local, since, for examples, $u_t = (1-D_x^2)^{-1}(uu_x)$. In [15] the authors proved that the BBM equation does not possess further conservation laws of this kind.

In [22] (see also [21]) a general theorem on conservation laws for arbitrary differential equation which does not require the existence of Lagrangians has been proved. This new theorem is based on the concept of adjoint equations for non-linear equations. There are many equations with physical significance are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations. In [21] Ibragimov generalized the concept of self-adjoint equations by introducing the definition of quasi-self-adjoint equations.

In this paper we study the Lie symmetries of equation

$$u_t + bu_x + a\left(u^m\right)_x + \left(u^n\right)_{xxt} = 0, \tag{1}$$

where a, b are constants and m or $n \neq 1$, by using the Lie method of infinitesimals. We determine, for equation (1), the subclasses of equations which are self-adjoint. We also determine, by using the notation and techniques of the work [21], [22], some nontrivial conservation laws for equation (1).

II. Lie Symmetries

To apply the classical method to Eq. (1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{array}{l} x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{array}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{2}$$

Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (2) leads to a set of twenty six determining equations. Solving this system we obtain $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta = \dfrac{\alpha(x,t)}{u^{n-1}} - \dfrac{k_1\, u}{2\, n} + \dfrac{\xi_x\, u}{2\, n}$ where ξ, τ and α are related by the following conditions:

$$\begin{array}{l} \xi_{xxx} n^2\, u^{2\,n} + k_1\, n\, u^{n+1} + 3\,\xi_x n\, u^{n+1} - k_1\, u^{n+1} \\ +\xi_x u^{n+1} - 2\,\alpha\, n^2\, u + 2\,\alpha\, n\, u = 0, \\ a\,\xi_{xx}\, m\, u^{m+n} + 2\,a\,\alpha_x n\, m\, u^m + b\,\xi_{xx} u^{n+1} \\ +2\alpha_{txx} n^2\, u^n + 2\alpha_x b\, n\, u + 2\alpha_t n\, u = 0, \\ -a\, k_1\, m^2\, u^{m+n} + a\,\xi_x m^2\, u^{m+n} + a\, k_1\, n\, m\, u^{m+n} \\ +2\,a\,\tau_t n\, m\, u^{m+n} + a\,\xi_x n\, m\, u^{m+n} \\ +2\,a\,\alpha\, n\, m^2\, u^m - 2\,a\,\alpha\, n^2\, m\, u^m + b\, k_1\, n\, u^{n+1} \\ +2\, b\tau_t n\, u^{n+1} \\ +b\,\xi_x\, n\, u^{n+1} - b\, k_1\, u^{n+1} + b\xi_x u^{n+1} \\ -2\,\alpha\, b\, n^2\, u + 2\,\alpha\, b\, n\, u = 0. \end{array}$$

The solutions of this system depend on the parameters of Eq. (1). If a and b are arbitrary constants, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t.$$

For $\lambda\mathbf{v}_1 + \mathbf{v}_2$ the similarity variables and similarity solution are:

$$\begin{array}{rcl} z & = & x - \lambda t \\ u & = & h(z) \end{array} \tag{3}$$

where $h(z)$ satisfies

$$\lambda(h^n)''' + \lambda h' - amh^{m-1}h' - bh' = 0.$$

This equation, after integrating once with respect to z, can be reduced to

$$\lambda\left(h^n\right)'' = ah^m + (b-\lambda)h + k_1, \tag{4}$$

where k_1 is an integrating constant.

The cases for which Eq.(1) with $b \neq 0$ have extra symmetries have been studied by Bruzón, Gandarias and Camacho in [8]:

Table 1: Symmetries for a Generalization of a Family of BBM Equations.

i	constants	V_3^i	$\mathbf{v}_\infty^i$
1	$a \cdot (m-1) = 0$	$(n-1)x\partial_x + (n-1)t\partial_t + 2u\partial_u$	
2	$m = 1,\, a = -b$	$(n-1)x\partial_x + 2u\partial_u$	$\tau(t)\partial_t$
3	$m = 2,\, n = 1$	$-t\partial_t + \left(u + \frac{b}{2a}\right)\partial_u$	
4	$m = 1,\, n = \frac{1}{2}$	$-x\partial_x + 4u\partial_u$	$\tau(t)\partial_t$

where $\tau(t)$ is arbitrary function.

III. Traveling wave solutions

Wang *et al* [32] introduced a method which is called the $\frac{G'}{G}$-expansion method to look for travelling wave solutions of nonlinear evolution equations. The main ideas of the proposed method are that the travelling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in $\frac{G'}{G}$, where $G = G(z)$ satisfies the linear second order ordinary differential equation (ODE) $G''(z) + \omega G'(z) + \zeta G(z) = 0$, the degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in a given nonlinear evolution equation, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

In [7] we consider the BBM equation (1) with $n = 1, m = 2$ and look for the travelling wave solution of this equation. In this case the reduced equation is

$$\lambda h'' = ah^2 + (b - \lambda)h + k_1. \tag{5}$$

We apply the $\frac{G'}{G}$-expansion method to equation (5). We suppose that solutions can be expressed by a polynomial in $\frac{G'}{G}$ in the form

$$h = \sum_{i=0}^{k} a_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $G = G(z)$ satisfies the linear second order ODE

$$G''(z) + \alpha G(z) + \beta G = 0, \tag{7}$$

a_i, $i = 0, \ldots, k$, α and β are constants to be determined later, $a_k \neq 0$.

General solutions of equation (7) are:

- If $\alpha^2 - 4\beta > 0$,

$$\begin{aligned} G(z) = \;& c_1 \cosh\left(\tfrac{z\alpha}{2} - \tfrac{1}{2} z\sqrt{\alpha^2 - 4\beta}\right) \\ & + c_2 \cosh\left(\tfrac{\alpha z}{2} + \tfrac{1}{2}\sqrt{\alpha^2 - 4\beta} z\right) \\ & - c_1 \sinh\left(\tfrac{z\alpha}{2} - \tfrac{1}{2} z\sqrt{\alpha^2 - 4\beta}\right) \\ & - c_2 \sinh\left(\tfrac{\alpha z}{2} + \tfrac{1}{2}\sqrt{\alpha^2 - 4\beta} z\right). \end{aligned} \tag{8}$$

- If $\alpha^2 - 4\beta < 0$,

$$\begin{aligned} G(z) = \;& \Big(c_2 \cos\left(\tfrac{1}{2} z\sqrt{4\beta - \alpha^2}\right) \\ & + c_1 \sin\left(\tfrac{1}{2} z\sqrt{4\beta - \alpha^2}\right)\Big) \\ & \left(\cosh\left(\tfrac{z\alpha}{2}\right) - \sinh\left(\tfrac{z\alpha}{2}\right)\right). \end{aligned} \tag{9}$$

- If $\alpha^2 = 4\beta$,

$$G(z) = (c_2 + c_1 z)\left(\cosh\left(\frac{z\alpha}{2}\right) - \sinh\left(\frac{z\alpha}{2}\right)\right). \tag{10}$$

By using (6) and (7) we obtain

$$h^2 = a_k^2 \left(\frac{G'}{G}\right)^{2k} + \cdots \tag{11}$$

$$h'' = k(k+1)a_k \left(\frac{G'}{G}\right)^{k+2} + \cdots \tag{12}$$

Considering the homogeneous balance between h'' and h^2 in (5), based on (11) and (12) we require that $k + 2 = 2k \Rightarrow k = 2$, so we can write (6) as

$$h = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \tag{13}$$

$a_2 \neq 0$.

Substituting the general solutions of (7) into (13) we obtain:

From (8),

$$\begin{aligned} h_1(z) = \;& \tfrac{a_2\alpha^2}{4} - \tfrac{a_1\alpha}{2} + a_0 + \\ & \tfrac{\sqrt{\alpha^2 - 4\beta}}{8}\left(-4a_2\alpha + 4a_1\right) H_1 \\ & + \tfrac{\alpha^2 - 4\beta}{8}\left(+2a_2\right)\left(H_1\right)^2, \end{aligned} \tag{14}$$

where $H_1(z) = \frac{c_2 \cosh(\frac{1}{2}\sqrt{\alpha^2 - 4\beta} z) + c_1 \sinh(\frac{1}{2}\sqrt{\alpha^2 - 4\beta} z)}{c_1 \cosh(\frac{1}{2}\sqrt{\alpha^2 - 4\beta} z) + c_2 \sinh(\frac{1}{2}\sqrt{\alpha^2 - 4\beta} z)}$.

From (9),

$$\begin{aligned} h_2(z) = \;& \tfrac{a_2\alpha^2}{4} - \tfrac{a_1\alpha}{2} + a_0 \\ & + \tfrac{2a_2}{8}\left(4\beta - \alpha^2\right)\left(H_2\right)^2 \\ & + \tfrac{-4a_2\alpha + 4a_1}{8}\sqrt{4\beta - \alpha^2}\left(H_2\right)^3, \end{aligned} \tag{15}$$

where $H_2(z) = \frac{c_2 \cos\left(\frac{1}{2} z\sqrt{4\beta - \alpha^2}\right) + c_1 \sin\left(\frac{1}{2} z\sqrt{4\beta - \alpha^2}\right)}{c_1 \cos\left(\frac{1}{2} z\sqrt{4\beta - \alpha^2}\right) - c_2 \sin\left(\frac{1}{2} z\sqrt{4\beta - \alpha^2}\right)}$.

From (10),

$$\begin{aligned} h_3(z) = \;& \tfrac{a_2\alpha^2}{4} - \tfrac{a_1\alpha}{2} + a_0 + \tfrac{(2a_2)c_1^2}{2(c_1 z + c_2)^2} \\ & + \tfrac{(-4a_2\alpha + 4a_1)c_1}{4(c_1 z + c_2)}. \end{aligned} \tag{16}$$

In the following we determine a_i, $i = 0, \ldots, 2$. From (13) we calculate h^2 and h'' and we substitute this expression in equation (5). Equating each coefficient of $\left(\frac{G'}{G}\right)^i$, $i = 0, \ldots, 2$ to zero, yields a set of simultaneous algebraic equations for $a_i, \alpha, \beta, \lambda$ and k_1. Solving this system, we obtain the set of solutions:

$$a_0 = \frac{\lambda\alpha^2 - b + 8\beta\lambda + \lambda}{2a}, \tag{17}$$

$$a_1 = \frac{6\alpha\lambda}{a}, \tag{18}$$

$$a_2 = \frac{6\lambda}{a}, \tag{19}$$

$$k_1 = -\frac{\left(\lambda\alpha^2 + b - 4\beta\lambda - \lambda\right)}{4a} \frac{\left(\lambda\alpha^2 - b - 4\beta\lambda + \lambda\right)}{4a}. \tag{20}$$

Substituting (17)-(19) into (14)-(16) we have three types of travelling wave solutions of the BBM equation (1) with $n = 1$ and $m = 2$:

If $\alpha^2 - 4\beta > 0$:

$$u_1(x,t) = \frac{3\left(\alpha^2 - 4\beta\right)\lambda}{2a} F_1^2 - \frac{2\lambda\alpha^2 + b - 8\beta\lambda - \lambda}{2a}, \tag{21}$$

where

$F_1 = \frac{c_2 \cosh\left(\frac{1}{2}\sqrt{\alpha^2 - 4\beta}(x - \lambda t)\right) + c_1 \sinh\left(\frac{1}{2}\sqrt{\alpha^2 - 4\beta}(x - \lambda t)\right)}{c_1 \cosh\left(\frac{1}{2}\sqrt{\alpha^2 - 4\beta}(x - \lambda t)\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{\alpha^2 - 4\beta}(x - \lambda t)\right)}$.

If $4\beta - \alpha^2 > 0$:

$$u_2(x,t) = \frac{3\left(4\beta - \alpha^2\right)\lambda}{2a} F_2^2 - \frac{2\lambda\alpha^2 + b - 8\beta\lambda - \lambda}{2a}, \tag{22}$$

where

$F_2 = \frac{c_1 \cos\left(\frac{1}{2}\sqrt{4\beta - \alpha^2}(x - \lambda t)\right) - c_2 \sin\left(\frac{1}{2}\sqrt{4\beta - \alpha^2}(x - \lambda t)\right)}{c_2 \cos\left(\frac{1}{2}\sqrt{4\beta - \alpha^2}(x - \lambda t)\right) + c_1 \sin\left(\frac{1}{2}\sqrt{4\beta - \alpha^2}(x - \lambda t)\right)}$.

If $\alpha^2 = 4\beta$:

$$u_3(x,t) = \frac{12\lambda c_1^2}{2a(c_2 + c_1(x - \lambda t))^2} + \frac{\lambda - b}{2a}. \tag{23}$$

IV. Exact Solutions

By making the change of variables

$$h^n = y \tag{24}$$

equation (4) becomes

$$\lambda y'' = a y^{\frac{m}{n}} + (b - \lambda) y^{\frac{1}{n}} + k_1. \tag{25}$$

After multiplying (25) by $2y'$ and integrating once with respect to z we get

$$\lambda(y')^2 = \frac{2an}{m+n} y^{\frac{m}{n}+1} + \frac{2(b-\lambda)n}{n+1} y^{\frac{1}{n}+1} + 2k_1 y + k_2, \quad (26)$$

where k_2 is an integrating constant.

Let us assume that equation (26) has solution of the form

$$y(z) = \alpha f^{\beta}(z), \quad (27)$$

where α and β are parameters to be determined later. By substituting (27) into (26) we obtain

$$\begin{aligned}(f')^2 = \; & \tfrac{2an}{(m+n)\lambda\alpha\beta^2} f^{\frac{\beta m}{n}-\beta+2} \\ & + \tfrac{2(b-\lambda)n}{(n+1)\lambda\alpha\beta^2} f^{\frac{\beta}{n}-\beta+2} \\ & + \tfrac{2k_1}{\alpha\beta^2\lambda} f^{-\beta+2} \\ & + \tfrac{k_2}{\alpha^2\beta^2\lambda} f^{-2\beta+2}.\end{aligned} \quad (28)$$

In the following we will determine the exponents and coefficients of equations (28). So that equation (28) is solvable in terms of Jacobi elliptic function, that is equation (28) becomes

$$(f')^2 = r + pf^2 + qf^4, \quad (29)$$

where r, p and q are constants.

Comparing the exponents and the coefficients of equations (28) and (29) we distinguish the following cases:

Case 1: If $k_1 = 0$ and $k_2 = 0$.

Subcase 1.1: $\beta = \dfrac{2}{m-1}$, $n = 1$ and $m \neq 1$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{4p\lambda}, \qquad a = \frac{(m+1)q(b-\lambda)}{2p}.$$

Subcase 1.2: $\beta = \dfrac{2m}{1-m}$ and $n = m$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{2m(m+1)q\lambda}, \qquad a = \frac{2mp(b-\lambda)}{(m+1)q}.$$

Subcase 1.3: $\beta = \dfrac{2n}{n-1}$ and $n = m$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{2m(m+1)r\lambda}, \qquad a = \frac{2mp(b-\lambda)}{(m+1)r}.$$

Subcase 1.4: $\beta = \dfrac{2n}{n-1}$ and $m = 2n-1$.

$$\alpha = \frac{(b-\lambda)(n-1)^2}{2n(n+1)r\lambda}, \qquad a = \frac{(3n-1)q(b-\lambda)}{(n+1)r}.$$

Subcase 1.5: $\beta = \dfrac{2}{1-m}$ and $n = 1$.

$$\alpha = \frac{(b-\lambda)(m-1)^2}{4p\lambda}, \qquad a = \frac{(m+1)r(b-\lambda)}{2p}.$$

Subcase 1.6: $\beta = \dfrac{2n}{n-1}$ and $m = 2n-1$.

$$\alpha = \frac{(n-1)^2(b-\lambda)}{2n(n+1)p\lambda}, \quad a = \frac{(3n-1)q(b-\lambda)}{(n+1)p\lambda}.$$

Subcase 1.7: $\lambda = b$, β is arbitrary and $m = n$.

$$\alpha \text{ is arbitrary}, \qquad a = \alpha\beta^2 pb.$$

Subcase 1.8: $\lambda = b$, $\beta = \frac{2n}{m-n}$.

$$\alpha = \frac{a(n-m)^2}{2bn(n+m)q}.$$

Subcase 1.9: $\lambda = b$, $\beta = \frac{2n}{n-m}$.

$$\alpha = \frac{a(n-m)^2}{2bn(n+m)r}.$$

Case 2: If $k_1 \neq 0$ and $k_2 = 0$.

Subcase 2.1: $\beta = 2$, $n = 1$ and $m = 2$.

$$\alpha = \frac{k_1}{2r\lambda}, \quad b = \frac{2k_1p}{r} + \lambda, \quad a = \frac{3qk_1}{r}.$$

Subcase 2.2: $\beta = -2$, $n = 1$ and $m = 2$.

$$\alpha = \frac{k_1}{2q\lambda}, \quad b = \frac{2k_1p}{q} + \lambda, \quad a = \frac{3k_1r}{q}.$$

Subcase 2.3: $\beta = -2$, $n = m = \frac{1}{2}$.

$$\alpha = \frac{k_1}{2q\lambda}, \quad b = \frac{3k_1r}{q} + \lambda, \quad a = \frac{2k_1p}{q}.$$

Subcase 2.4: $\beta = 2$, $n = m = \frac{1}{2}$.

$$\alpha = \frac{k_1}{2r\lambda}, \quad b = \frac{3k_1q}{r} + \lambda, \quad a = \frac{2k_1p}{r}.$$

Case 3: If $k_1 = 0$ and $k_2 \neq 0$.

Subcase 3.1: $\beta = 1$, $n = 1$ and $m = 3$.

$$\alpha = \pm\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}, \quad b = \lambda\left[1 \pm p\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}\right],$$

$$a = \pm 2q\lambda\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}.$$

Subcase 3.2: $\beta = -1$, $n = 1$ and $m = 3$.

$$\alpha = \pm\left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}, \quad b = \lambda\left[1 \pm p\left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}\right],$$

$$a = \pm 2r\lambda\left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}.$$

Subcase 3.3: $\beta = 1$, $n = m = \frac{1}{3}$.

$$\alpha = \pm\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}, \quad b = \lambda\left[1 \pm 2q\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}\right],$$

$$a = \pm p\lambda\left(\frac{k_2}{r\lambda}\right)^{\frac{1}{2}}.$$

Subcase 3.4: $\beta = -1$, $n = m = \frac{1}{3}$.

$$\alpha = \pm\left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}, \quad b = \lambda\left[1 \pm 2r\left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}\right],$$

$$a = \pm p\lambda \left(\frac{k_2}{q\lambda}\right)^{\frac{1}{2}}.$$

Since in all these cases, r, p and q are arbitrary constants, we may choose them properly such that the corresponding solution f of the ODE (29) are expressed in terms of the Jacobian elliptic functions. In the following we present some exact solutions.

- If $r = 1$, $p = -(1 + c^2)$, $q = c^2$, then

$$y = \alpha\,(\mathrm{sn}(z|c))^{\beta},$$

where $\mathrm{sn}(z|c)$ is the Jacobi elliptic function, is a solution of equation (26), [2].

From Subcase 3.1 for $\lambda = k_2$, $n = 1$, $m = 3$, $a = 2k_2c^2$ and $b = -k_2c^2$ we obtain the particular solution of equation (26)

$$y = \mathrm{sn}(z|c).$$

From (24) and (3) for $c = 1$, $n = 1$, $m = 3$ and $a = -2b$ we obtain the exact solution of (1) given by

$$u(x,t) = \tanh(x + bt). \tag{30}$$

If $b = -\frac{1}{2}$, (30) describes a kink solution (see Fig.1).

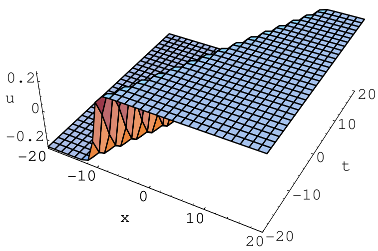

Fig. 1. Solution (30) for $b = -\frac{1}{2}$.

From Subcase 2.4 for $\lambda = \frac{k_1}{2}$, $a = -2k_1(c^2 + 1)$ and $b = k_1(3c^2 + \frac{1}{2})$ we obtain the solution of equation (26)

$$y = \mathrm{sn}^2(z|c).$$

From (24) and (3), for $c = 0$, $m = n = \frac{1}{2}$ and $a = -4b$, to yield

$$u(x,t) = \sin^4(x - bt). \tag{31}$$

- If $r = \frac{1-c^2}{4}$, $p = \frac{1+c^2}{2}$, $q = \frac{1-c^2}{4}$, $f = \mathrm{nc}(z|c) \pm \mathrm{sc}(z|c)$ is solution of equation (29), [2]. Then

$$y = \alpha\,[\mathrm{nc}(z|c) \pm \mathrm{sc}(z|c)]^{\beta}$$

is solution of equation (26), where α and β are arbitrary functions, $\mathrm{nc}(z|c) = \frac{1}{\mathrm{cn}(z|c)}$, $\mathrm{sc}(z|c) = \frac{\mathrm{sn}(z|c)}{\mathrm{cn}(z|c)}$ where $\mathrm{sn}(z|c)$ and $\mathrm{cn}(z|c)$ are the first and the second Jacobian elliptic functions, respectively (the elliptic sine and the elliptic cosine).

From Subcase 1.7 for $\lambda = b$, $a = b\beta^2$ and $n = m$ we obtain the particular solution of equation (26)

$$y = [\mathrm{nc}(z|1) \pm \mathrm{sc}(z|1)]^{\beta}.$$

From (24) and (3) if $m = n$ and $a = b\beta^2$ we obtain the solution of equation (1)

$$u(x,t) = [\cosh(x - bt) \pm \sinh(x - bt)]^{\beta}. \tag{32}$$

- If $p = 1$ and $q = -1$,

$$y = \alpha\,(\mathrm{cn}(z|1))^{\beta}$$

is solution of equation (26).

From subcase 1.1 for $\lambda = \dfrac{b(m-1)^2}{m^2 - 2m + 5}$, $n = 1$ and $a = -\dfrac{2b(m+1)}{m^2 - 2m + 5}$, the solution of equation (26) is

$$y = \mathrm{sech}^{\frac{2}{m-1}}(z).$$

From (24) and (3) we obtain the solution of equation (1)

$$u(x,t) = \mathrm{sech}^{\frac{2}{m-1}}(x - \lambda t). \tag{33}$$

For $m = 2$ and $\lambda = 1$, (33) describes a soliton moving along a line with constant velocity (see Fig.2).

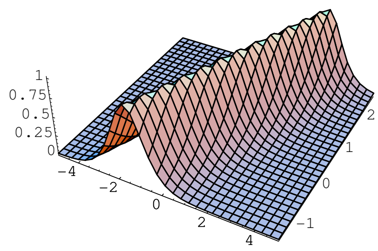

Fig. 2. Solution (33) for $m = 2$, $\lambda = 1$ and $a = -6$.

Solutions (30) and (33) were first found in [26]. As far as we know, solutions (31) and (32) are new and have not been previously described in the literature.

V. Determination of self-adjointness equations

Given (1+1)-dimensional evolution equation of order n, $F \equiv F(\mathrm{x}, u, \mathbf{u}^{(1)}(\mathrm{x}), \ldots, \mathbf{u}^{(n)}(\mathrm{x})) = 0$, where $\mathrm{x} = (x,t)$ are independent variables, $u = u(\mathrm{x})$ is a dependent variable and $\mathbf{u}^{(l)}(\mathrm{x})$ denotes the set of all the partial derivatives of order l of u; a conservation law is of the form

$$D_t\,\rho + D_x\,J = 0,$$

where ρ is the conserved density, J is the associated flux, $D_xJ = \dfrac{\partial J}{\partial x} + \displaystyle\sum_{k=0}^{N} \dfrac{\partial J}{\partial u_{kx}} u_{(k+1)x}$, N is the order of J, and $D_t\rho = \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{k=0}^{M} \dfrac{\partial \rho}{\partial u_{kx}} D_x^k u_t$, with M the order of ρ.

In [22] Ibragimov introduced a new theorem. The theorem is valid for any system of differential equations where the number of equations is equal to the number of dependent variables. The new theorem does not require existence of a Lagrangian and this theorem is based on a concept of an adjoint equation for non-linear equations.

Given

$$F = u_t + bu_x + a\left(u^m\right)_x + \left(u^n\right)_{xxt},$$

the adjoint equation $F^* = 0$ is defined

$$F^* \equiv \frac{\delta}{\delta u}(vF) = 0,$$

where $v = v(x,t)$ is a new dependent variable and the variational derivative is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i\left(\frac{\partial}{\partial u_i}\right) + D_i D_k\left(\frac{\partial}{\partial u_{ik}}\right)$$
$$-D_i D_j D_k\left(\frac{\partial}{\partial u_{ijk}}\right) + \cdots$$

We obtain F^*. Setting $v = u$, $F^* \equiv -a\,m\,u^{m-1}\,u_x - b\,u_x - n\,u^{n-1}\,u_{t\,x\,x} - u_t$. Comparing F^* with F we obtain that $F^* = \lambda\,F$ if $\lambda = -1$ and $n = 1$ and, consequently, we get the following result:

Proposition. Equation $F \equiv u_t + bu_x + a\left(u^m\right)_x + \left(u^n\right)_{xxt} = 0$ is self-adjoint if $n = 1$, i.e. when it has the following form

$$F = u_t + bu_x + a\left(u^m\right)_x + u_{xxt}.$$

VI. Conservation laws

Consider now the nonlocal conservation theorem method given by Ibragimov [21], [22]. The formal Lagragian is

$$\mathcal{L} = v(u_t + bu_x + a\left(u^2\right)_x + u_{xxt}). \tag{34}$$

Conservation law $D_t(C^1) + D_x(C^2) = 0$ corresponding to an operator

$$\mathbf{v} = \xi^1\partial_t + \xi^2\partial x + \eta\partial u$$

is given by

$$\begin{aligned} C^1 = \;& \xi^1\mathcal{L} + W\,(\frac{\partial\mathcal{L}}{\partial u_t}) + D_{xx}\left(\frac{\partial\mathcal{L}}{\partial u_{txx}}\right) \\ & + D_x(W)\left(-D_x\left(\frac{\partial\mathcal{L}}{\partial u_{txx}}\right)\right) \\ & + D_{xx}(W)\frac{\partial\mathcal{L}}{\partial u_{txx}} \\ C^2 = \;& \xi^2\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right)\right. \\ & + D_{xt}\left(\frac{\partial\mathcal{L}}{\partial u_{xxt}}\right) + D_{xt}\left(\frac{\partial\mathcal{L}}{\partial u_{xtx}}\right) \\ & \left. + D_{xx}\left(\tfrac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right] \\ & + D_t(W)\left[-D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xtx}}\right) - D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xtt}}\right)\right] \\ & + D_x(W)\left[\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right) - D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xxt}}\right)\right. \\ & \left. - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right] + D_{xx}(W)\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) \\ & + D_{xt}(W)\left(\frac{\partial\mathcal{L}}{\partial u_{xtx}}\right) + D_{xt}(W)\left(\frac{\partial\mathcal{L}}{\partial u_{xxt}}\right) \end{aligned}$$

where

$$W = \eta - \xi^1\,u_t - \xi^2\,u_x.$$

BBM equation for $n = 1$ and $m = 2$ admits the following generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = -t\partial t + \tfrac{b}{2a}\,u\partial u,$$

From generators $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain trivial conservation laws. For generator

$$\mathbf{v}_3 = -t\partial t + \frac{b}{2a}\,u\partial u,$$

the normal form for this group is

$$W = t\,u_t + u + \frac{b}{2\,a}. \tag{35}$$

The vector components are

$$\begin{aligned} C^1 \;=\;& \frac{tu_tv_{xx}}{3} + \frac{uv_{xx}}{3} + \frac{bv_{xx}}{6a} - \frac{u_xv_x}{3} - \frac{tu_{tx}v_x}{3} \\ & + \frac{u_{xx}v}{3} - 2atuu_xv - btu_xv - \frac{2tu_{txx}v}{3} \\ & + uv + \frac{bv}{2a} \end{aligned} \tag{36}$$

$$\begin{aligned} C^2 \;=\;& -\frac{tu_{tt}v_x}{3} - \frac{2u_tv_x}{3} + \frac{2tu_tv_{tx}}{3} + \frac{2uv_{tx}}{3} \\ & + \frac{bv_{tx}}{3a} - \frac{u_xv_t}{3} - \frac{tu_{tx}v_t}{3} + \frac{2tu_{ttx}v}{3} \\ & + \frac{4u_{tx}v}{3} + 2atuu_tv + btu_tv + 2au^2v \end{aligned} \tag{37}$$

$$+2buv+\frac{b^2v}{2a}$$

Substituting in (36) and (37) the expression (34) for $\mathcal{L}$ and the normal form (35) for W, we get

$$\begin{aligned} C^1 &= \frac{tu_tv_{xx}}{3}+\frac{uv_{xx}}{3}+\frac{bv_{xx}}{6a}-\frac{u_xv_x}{3}-\frac{tu_{tx}v_x}{3} \quad (38)\\ &+\frac{u_{xx}v}{3}-2atuu_xv-btu_xv-\frac{2tu_{txx}v}{3}+uv\\ &+\frac{bv}{2a}\\ C^2 &= -\frac{tu_{tt}v_x}{3}-\frac{2u_tv_x}{3}+\frac{2tu_tv_{tx}}{3}+\frac{2uv_{tx}}{3} \quad (39)\\ &+\frac{bv_{tx}}{3a}-\frac{u_xv_t}{3}-\frac{tu_{tx}v_t}{3}\\ &+\frac{2tu_{ttx}v}{3}+\frac{4u_{tx}v}{3}+2atuu_tv+btu_tv\\ &+2au^2v+2buv+\frac{b^2v}{2a}\end{aligned}$$

Now we substitute in (38) and (39) v for u,

$$\begin{aligned} C^1 = &\ \frac{t\,u_t\,u_{x\,x}}{3}+\frac{2\,u\,u_{x\,x}}{3}+\frac{b\,u_{x\,x}}{6\,a}-\frac{(u_x)^2}{3}-\frac{t\,u_{t\,x}\,u_x}{3}\\ &-2\,a\,t\,u^2\,u_x-b\,t\,u\,u_x-\frac{2\,t\,u\,u_{t\,x\,x}}{3}\\ &+u^2+\frac{b\,u}{2\,a}\\ C^2 = &\ -\frac{t\,u_{t\,t}\,u_x}{3}-u_t\,u_x+\frac{2\,t\,u\,u_{t\,t\,x}}{3}+\frac{t\,u_t\,u_{t\,x}}{3}\\ &+2\,u\,u_{t\,x}+\frac{b\,u_{t\,x}}{3\,a}+2\,a\,t\,u^2\,u_t\\ &+b\,t\,u\,u_t+2\,a\,u^3+2\,b\,u^2+\frac{b^2\,u}{2\,a},\end{aligned}$$

shift the terms of the form $D_x(\cdots)$ into C_2 and finally arrive at the conserved vector with the following components:

$$C^1=-\left(u_x\right)^2+u^2+\frac{b\,u}{2\,a}$$

$$C^2=2\,u\,u_{t\,x}+\frac{b\,u_{t\,x}}{2\,a}+\frac{4\,a\,u^3}{3}+\frac{3\,b\,u^2}{2}+\frac{b^2\,u}{2\,a}$$

VII. CONCLUSIONS

We have considered classical symmetries of a $B(m,n)$ equation. In the case $n=1$ and $m=2$, by using the $\frac{G'}{G}$-expansion method, we have obtained three types of travelling wave solutions. We obtain for special values of the parameters of this equation, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic function solutions and their degenerative solutions (soliton, kink and compactons). The concept of self-adjoint equation was introduced by NH Ibragimov in [21], [22]. In this paper we found the general classes of the self equations (1). By using the Ibragimovs Theorem on conservation laws, we have derived conservation laws for this equation.

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