On Weighted Possibilistic Informational Coefficient of Correlation

Robert Fullér, István Á. Harmati, Péter Várlaki, Imre Rudas

***Abstract*—In their previous works Fullér et al. introduced the notions of weighted possibilistic correlation coefficient and correlation ratio as measures of dependence between possibility distributions (fuzzy numbers). In this paper we introduce a new measure of strength of dependence between marginal possibility distributions, which is based on the informational coefficient of correlation. We will show some examples that demonstrate some good properties of the proposed measure.**

***Index Terms*—Fuzzy number, Possibility distribution, Measure of dependence, Mutual information, Correlation.**

I. INTRODUCTION

UNCERTAIN informations can be diveded into two main categories: incomplete and imprecise information. Probability distributions can be interpreted as carriers of incomplete information [1], and possibility distributions can be interpreted as carriers of imprecise information. Measuring dependence between uncertain variables plays a fundamental role in both categories. In probability theory there are well-known measures of dependence between random variables, such as correlation coefficient, correlation ratio, mean square contingency, and so on. In possibility theory there are several treatments for characterizing dependence between fuzzy numbers, see for example [2] and [3].

In this approach we use simple probability distributions to build up measures of dependence between possibility distributions. Namely, we equip each level set of a possibility distribution with a uniform probability distribution, then determine a probabilistic measure of dependence, and then define measures on possibility distributions by integrating these weighted probabilistic notions over the set of all membership grades [4], [5]. These weights (or importances) can be given by weighting functions.

Definition 1. *A function $g\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if g is non-negative, monotone increasing and satisfies the*

$$\int_0^1 g(\gamma)\,\mathrm{d}\gamma = 1$$

normalization condition

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In other words a possibilistic measure of dependence is the g-weighted average of the probabilistic measure of depenence. Different weighting functions can give different (case-dependent) importances to level-sets of possibility istributions. We should note here that the choice of uniform probability distribution on the level sets of possibility distributions is not without reason. We suppose that each point of a given level set is equally possible and then we apply Laplace's principle of Insufficient Reason: if elementary events are equally possible, they should be equally probable (for more details and generalization of principle of Insufficient Reason see [6], page 59). The uniform distribution is not the only way, for example we can deal with probabilty distributions whose density function is similar (has the same shape) to the joint possibility distribution, see [7].

Definition 2. *A* fuzzy number A *is a fuzzy set* $\mathbb{R}$ *with a normal, fuzzy convex and continuous membership function of bounded support.*

Fuzzy numbers can be considered as possibility distributions. A fuzzy set C in $\mathbb{R}^2$ is said to be a joint possibility distribution of fuzzy numbers A, B, if it satisfies the relationships

$$\max\{x \mid C(x,y)\} = B(y),$$

and

$$\max\{y \mid C(x,y)\} = A(x),$$

for all $x, y \in \mathbb{R}$. Furthermore, A and B are called the marginal possibility distributions of C. Marginal possibility distributions are always uniquely defined by their joint possibility distribution by the principle of falling shadows. A γ-level set (or γ-cut) of a fuzzy number A is a non-fuzzy set denoted by $[A]^\gamma$ and defined by

$$[A]^\gamma = \{t \in X \mid A(t) \geq \gamma\},$$

if $\gamma > 0$ and $\mathrm{cl}(\mathrm{supp}A)$ if $\gamma = 0$, where $\mathrm{cl}(\mathrm{supp}A)$ denotes the closure of the support of A.

II. FORMER LEVEL-BASED MEASURES OF CORRELATION

A. Correlation Coefficient

In probability theory the correlation coefficient of random variables X and Y is defined by

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{E(XY) - E(X)\cdot E(Y)}{D(X)\cdot D(Y)},$$

where $E(X)$, $E(Y)$ and $E(XY)$ are expected value os X, Y and $X \cdot Y$ respectively, and $D(X)$, $D(Y)$ are the square roots of the variances.

In 2011 Fullér, Mezei and Várlaki introduced the principle of correlation (see [8]) that improves the earlier definition introduced by Carlsson, Fullér and Majlender in 2005 (see [4]). The main drawback of the earlier definition that it does not necessarily take its values from $[-1;1]$ if some level sets of the joint possibility distribution are not convex.

Definition 3. *The g-weighted possibilistic correlation coefficient of fuzzy numbers A and B (with respect to their joint distribution C) is defined by*

$$\rho_g(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) g(\gamma)\,\mathrm{d}\gamma,$$

where

$$\rho(X_\gamma, Y_\gamma) = \frac{\operatorname{cov}(X_\gamma, Y_\gamma)}{\sqrt{\operatorname{var}(X_\gamma)}\sqrt{\operatorname{var}(Y_\gamma)}},$$

and, where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$, and $\operatorname{cov}(X_\gamma, Y_\gamma)$ denotes their probabilistic covariance.

If A and B are non-interactive fuzzy numbers then their joint possibility distribution is defined by $C = A \times B$, so we the membership function of the joint possibiliy distribution is determined from the membership functions of A and B by the $\min$ operator. A more general notion of independence of fuzzy numbers can be found in [9]. Since all $[C]^\gamma$ are rectangular and the probability distribution on $[C]^\gamma$ is defined to be uniform we get $\operatorname{cov}(X_\gamma, Y_\gamma) = 0$, for all $\gamma \in [0,1]$. So the g-weighted possibilistic covariance $\operatorname{cov}_g(A,B) = 0$ and the g-weighted possibilistic correlation coefficient $\rho_g(A,B) = 0$ for any weighting function g. That is, non-interactivity entails zero correlation.

Zero correlation does not always implies non-interactivity. Let A, B be fuzzy numbers, let C be their joint possibility distribution, and let $\gamma \in [0,1]$. Suppose that $[C]^\gamma$ is symmetrical, i.e. there exists $a \in \mathbb{R}$ such that

$$C(x,y) = C(2a - x, y),$$

for all $x, y \in [C]^\gamma$ (*the line defined by $\{(a,t) | t \in \mathbb{R}\}$ is the axis of symmetry of $[C]^\gamma$*). In this case $\operatorname{cov}(X_\gamma, Y_\gamma) = 0$ and $\rho_g(A,B) = 0$ for any weighting function g. (see [10]). For more example on nonzero correlation see [11], [12] and [13].

B. Correlation Ratio

In statistics, the correlation ratio is a measure of the relationship between the statistical dispersion within individual categories and the dispersion across the whole population or sample. The correlation ratio was originally introduced by Karl Pearson [14] as part of analysis of variance and it was extended to random variables by Andrei Nikolaevich Kolmogorov [15] as a square root of

$$\eta^2(X|Y) = \frac{D^2[E(X|Y)]}{D^2(X)},$$

where X and Y are random variables. If X and Y have a joint probability density function, denoted by $f(x,y)$, then we can compute $\eta^2(X|Y)$ using the following formulas

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f(x|y)\,\mathrm{d}x$$

and

$$D^2[E(X|Y)] = E(E(X|Y) - E(X))^2,$$

and where,

$$f(x|y) = \frac{f(x,y)}{f(y)}.$$

It measures a functional dependence between random variables X and Y. It takes on values between 0 (no functional dependence) and 1 (purely deterministic dependence).

In 2010 Fullér, Mezei and Várlaki introduced the definition of possibilistic correlation ratio for marginal possibility distributions (see [16]).

Definition 4. *Let us denote A and B the marginal possibility distributions of a given joint possibility distribution C. Then the* g-weighted possibilistic correlation ratio *of marginal possibility distribution A with respect to marginal possibility distribution B is defined by*

$$\eta_g^2(A|B) = \int_0^1 \eta^2(X_\gamma | Y_\gamma) g(\gamma)\,\mathrm{d}\gamma$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$, and $\eta^2(X_\gamma|Y_\gamma)$ denotes their probabilistic correlation ratio.

III. Rényi's Postulates for Measures of Dependence

We use measures of dependence between random variables to determine measures of dependence between possibility distributions, so it is natural to describe what conditions should satisfy a good measure. In [17] A. Rényi gave seven postulates which should be fulfilled by a suitable measure of dependence between random variables X and Y ($\delta(X,Y)$):

A) $\delta(X,Y)$ is defined for any pair of random variables X and Y, neither of them being constant with probability 1.
B) $\delta(X,Y) = \delta(Y,X)$.
C) $0 \leq \delta(X,Y) \leq 1$.
D) $\delta(X,Y) = 0$ if and only if X and Y are independent.
E) $\delta(X,Y) = 1$, if there is a strict dependence between X and Y, i.e. either $X = g(Y)$ or $Y = f(X)$, where $g(x)$ and $f(x)$ are Borel-measurable functions.
F) If the Borel-measurable functions $f(x)$ and $g(x)$ maps the real axis in a one-to-one way ono itself, $\delta(f(X), g(Y)) = \delta(X,Y)$.
G) If the joint distribution of X and Y is normal, then $\delta(X,Y) = |\rho(X,Y)|$, where $\rho(X,Y)$ is the correlation coefficient of X and Y.

The correlation coefficient of the random variables X and Y is defined only if $D(X)$ and $D(Y)$ are finite and positve. It may be zero also if X and Y are not independent, moreover, it may vanish inspite of functional dependence between X and Y. For example if the distribution of X is symmetrical to zero and $Y = X^2$, then $\rho(X,Y) = 0$. $|\rho(X,Y)| = 1$ is equal to 1 if and only if there is a linear relationship between X and Y. The correlation coefficient satisfies postulates B and G, and its absolute value satisfies B, C and G.

TABLE I
MEASURES OF DEPENDENCE AND FULFILLMENT OF THE POSTULATES

	A	B	C	D	E	F	G
correlation coefficient		✓					✓
absolute value of corr. coeff.		✓	✓				✓
correlation ratio			✓		✓		✓
symmetrical correlation ratio		✓	✓		✓		✓
maximal correlation	✓	✓	✓	✓	✓	✓	✓
mutual information	✓	✓		✓		✓	
informational coeff. of corr.	✓	✓	✓	✓	✓	✓	✓

The correlation ratio is defined provided that $D(Y)$ exist and is positive. It is not symmetric, but one can consider instead of $\eta(X|Y)$ the quantity

$$\max\{\eta(X|Y), \eta(Y|X)\}$$

which is symmetric. The correlation ratio satisfies postulates B, C, E and G, the symmatric correlation ratio satisfies B, C, D, E and G.

The maximal correlation introduced by H. Gebelein ([18]) of random variables X and Y is defined by

$$S(X,Y) = \sup_{f,g}\{\rho(f(X), g(Y))\}$$

where $\rho(\cdot,\cdot)$ is the correlation coefficient, and $f(x)$ and $g(x)$ run over all Borel-measurable functions such that $f(X)$ and $g(Y)$ have finite and nonzero variance. It has all the properties A to G listed above, but unfortunately the maximal correlation is very difficult to determine and there does not always exist such functions $f_0(x)$ and $g_0(x)$ that

$$S(X,Y) = \rho(f_0(X), g_0(Y)) .$$

IV. INFORMATIONAL MEASURE OF CORRELATION

The well-known measures do not satisfy Rényi's postulates. For a better candidate first we recall the definition of mutual information:

Definition 5. *For any two continous random variables X and Y (admitting a joint probability density), their mutual information is given by*

$$I(X,Y) = \int\int f(x,y)\ln\frac{f(x,y)}{f_1(x)\cdot f_2(y)}\mathrm{d}x\mathrm{d}y$$

where $f(x,y)$ is the joint probability density function of X and Y, and $f_1(x)$ and $f_2(y)$ are the marginal density functions of X and Y, respectively.

Some properties of the mutual information:

- $I(X,Y) \geq 0$.
- $I(X,Y) = 0$ if and only if X and Y are independent.
- $I(X,Y) = \infty$ if X and Y are conmtinous and there is a functional relationship between X and Y.

If the joint probability distribution is uniform on the γ-levels, then the joint density function on a γ-level ($f(x,y)$) is constant, so the formula above will be simpler. Let denote T_γ the area of the γ-level. Then

$$\begin{aligned} I(X,Y) &= \int\int\frac{1}{T_\gamma}\ln\frac{\frac{1}{T_\gamma}}{f_1(x)f_2(y)}\mathrm{d}x\mathrm{d}y \\ &= \int\int\frac{1}{T_\gamma}\ln\frac{1}{T_\gamma}\mathrm{d}x\mathrm{d}y - \int\int\frac{1}{T_\gamma}\ln f_1(x)\,\mathrm{d}y\mathrm{d}x \\ &- \int\int\frac{1}{T_\gamma}\ln f_2(y)\,\mathrm{d}x\mathrm{d}y = \\ &= \ln\frac{1}{T_\gamma} - \int f_1(x)\ln f_1(x)\,\mathrm{d}x - \int f_2(y)\ln f_2(y)\,\mathrm{d}y\,. \end{aligned}$$

Moreover, if the joint probability distribution is uniform on the γ-levels, and the marginal random variables X and Y has the same distribution, then

$$I(X,Y) = \ln\frac{1}{T_\gamma} - 2\int f_1(x)\ln f_1(x)\,\mathrm{d}x\,.$$

Easy to check that the mutual information satisfies Rényi's postulates except C, E and G. Based on the mutual information Linfoot ([19]) introduced a measure of dependence, which satisfies all of the postulates.

Definition 6. *For two random variables X and Y, let denote $I(X,Y)$ the mutual information between X and Y. Their informational coefficient of correlation is given by*

$$L(X,Y) = \sqrt{1 - e^{-2I(X,Y)}}\,.$$

Based on the definition above, we can define the following:

Definition 7. *Let us denote A and B the marginal possibility distributions of a given joint possibility distribution C. Then the* g-weighted possibilistic informational coefficient of correlation *of marginal possibility distributions A and B is defined by*

$$L(A,B) = \int_0^1 L(X_\gamma, Y_\gamma)g(\gamma)\,\mathrm{d}\gamma$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0,1]$, and $L(X_\gamma, Y_\gamma)$ denotes informational coefficient of correlation.

V. EXAMPLES

First we show that non-interactivity implies zero for informational coefficient of correlation. In the second we show an example for non-zero correlation. Then we give two examples when the correlation coefficients and the correlation ratios are also zero, but the marginal distributions are not independent. So in these cases the correlation coefficient and the correlation ratio are not appropriate tools for measuring the dependence, but this problem not arises with the informational coefficient of correlation. Finally we show an example when the measures of correlation depend on the γ-level sets.

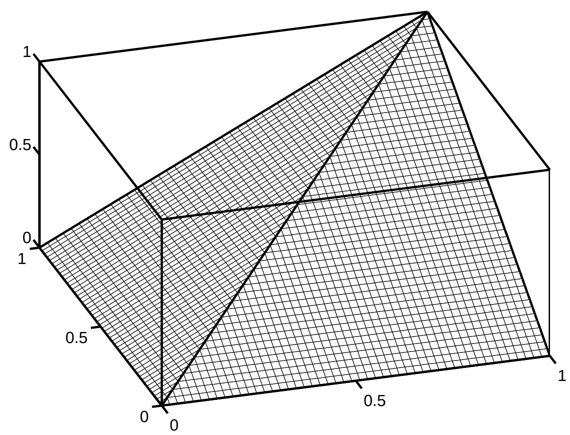

Fig. 1. Joint possibility distribution defined by the $\min$ operator (Mamdani t-norm). This is the case of non-interactivity of A and B, all of the γ-level sets are rectangulars.

A. Non-interactivity implies zero correlation

The joint possibility distribution is defined by the Mamdani t-norm ([20]), see Fig.1:

$$C(x,y) = \begin{cases} \min\{x,y\} & \text{if } 0 \le x, y \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

The marginal possibility distributions are

$$A(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

$$B(y) = \begin{cases} y & \text{if } 0 \le y \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

In this case the γ-level set is a square, with vertices $(0,0)$, $(0,1-\gamma)$, $(1-\gamma,0)$ and $(1-\gamma,1-\gamma)$. This is the case of non-interactivity of the marginal possibility distributions. The area of the γ-level set is

$$T_\gamma = (1-\gamma)^2 \, .$$

The joint density function is:

$$f(x,y) = \begin{cases} \dfrac{1}{T_\gamma} & \text{if } \gamma \le x, y \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

The marginal density function (we have the same expression for $f_2(y)$ with y) is:

$$f_1(x) = \begin{cases} \dfrac{1}{T_\gamma}(1-\gamma) & \text{if } \gamma \le x \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

In this case X_γ and Y_γ are independent (in probability sense), because:

$$f_1(x) \cdot f_2(y) = \frac{1}{T_\gamma}(1-\gamma) \cdot \frac{1}{T_\gamma}(1-\gamma) = \frac{1}{(1-\gamma)^2} = \frac{1}{T_\gamma} = f(x,y) \, .$$

So $I(X_\gamma, Y_\gamma) = 0$, and then the informational coefficient of correlation is

$$L(X_\gamma, Y_\gamma) = \sqrt{1 - e^{-2I(X_\gamma, Y_\gamma)}} = 0 \, .$$

In this case the correlation measures are zeros for all γ, so the g-weighted measures of correlation between the marginal possibility distributions A and B for arbitrary weighting function $g(\gamma)$:

$$L_g(A,B) = \int_0^1 L(X_\gamma, Y_\gamma) \cdot g(\gamma) \, \mathrm{d}\gamma = 0$$

$$\rho_g(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma) \cdot g(\gamma) \, \mathrm{d}\gamma = 0$$

$$\eta_g(A,B) = \int_0^1 \eta(X_\gamma, Y_\gamma) \cdot g(\gamma) \, \mathrm{d}\gamma = 0$$

B. Łukasiewitz t-norm

The joint possibility distribution is defined by the well-known Łukasiewitz t-norm ([21]), see Fig.2:

$$C(x,y) = \begin{cases} \max\{x+y-1, 0\} & \text{if } 0 \le x, y \le 1 \\ & \text{and } x+y \ge 1 \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

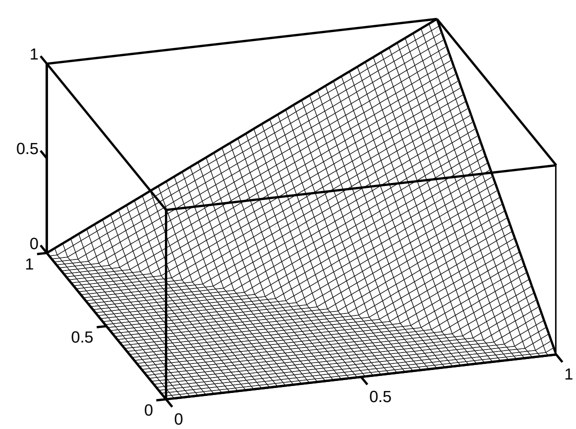

Fig. 2. Joint possibility distribution defined by the Łukasiewitz t-norm. In this case the γ-level sets are similar triangles.

The marginal possibility distributions are

$$A(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

$$B(y) = \begin{cases} y & \text{if } 0 \le y \le 1 \ , \\ 0 & \text{otherwise} \, . \end{cases}$$

The γ-level set:

$$[C]^\gamma = \left\{ (x,y) \in \mathbb{R}^2 | \ \gamma \le x, y \le 1, \ x+y \ge 1+\gamma \right\} \, .$$

The joint probability distribution on $[C]^\gamma$:

$$f(x,y) = \begin{cases} \dfrac{1}{T_\gamma} & \text{if } (x,y) \in [C]^\gamma \ . \\ 0 & \text{otherwise} \, . \end{cases}$$

where T_γ is the area of the γ-level set:

$$T_\gamma = \frac{(1-\gamma)^2}{2}.$$

The marginal density function (we have the same expression for $f_2(y)$ with y):

$$f_1(x) = \begin{cases} \dfrac{1}{T_\gamma}(x-\gamma) & \text{if } \gamma \le x \le 1 \ , \\ 0 & \text{otherwise}\,. \end{cases}$$

For the mutual information $I(X_\gamma, Y_\gamma)$ we have to compute:

$$\int_\gamma^1 f_1(x) \ln f_1(x)\,\mathrm{d}x = \int_\gamma^1 \frac{1}{T_\gamma}(x-\gamma) \ln \frac{1}{T_\gamma}(x-\gamma)\,\mathrm{d}x$$

$$= \left[t = \frac{x-\gamma}{T_\gamma}\right] = \int_0^{\frac{1-\gamma}{T_\gamma}} t \ln t \cdot T_\gamma \,\mathrm{d}t$$

$$= T_\gamma \cdot \left[\frac{t^2}{2}\ln t - \frac{t^2}{4}\right]_0^{\frac{1-\gamma}{T_\gamma}} = \ln 2 - \ln(1-\gamma) - \frac{1}{2}.$$

With this result the mutual information is

$$\begin{aligned} & I(X_\gamma, Y_\gamma) = \\ & = \ln\frac{1}{T_\gamma} - \int f_1(x)\ln f_1(x)\,\mathrm{d}x - \int f_2(y)\ln f_2(y)\,\mathrm{d}y \\ & = \ln\frac{1}{T_\gamma} - 2\cdot\left(\ln 2 - \ln(1-\gamma) - \frac{1}{2}\right) \\ & = \ln\frac{2}{(1-\gamma)^2} - 2\cdot\left(\ln 2 - \ln(1-\gamma) - \frac{1}{2}\right) = \\ & = 1 - \ln 2\,. \end{aligned}$$

From this the informational coefficient of correlation:

$$\begin{aligned} L(X_\gamma, Y_\gamma) &= \sqrt{1 - e^{-2I(X_\gamma, Y_\gamma)}} = \sqrt{1 - e^{-2(1-\ln 2)}} \\ &= \sqrt{1 - 4e^{-2}} \approx 0.6772\,. \end{aligned}$$

For this possibility distribution the correlation coefficient is (see [23])

$$\rho(X_\gamma, Y_\gamma) = -\frac{1}{2}$$

and the correlation ratio is

$$\eta^2(X_\gamma, Y_\gamma) = \frac{1}{4} \quad \Rightarrow \quad \eta(X_\gamma, Y_\gamma) = \frac{1}{2}$$

In this example $\eta^2 = \rho^2$, because of the linear relationship between X_γ and Y_γ. In this case the correlation measures are not depend on the level γ, so the g-weighted measures of correlation between the marginal possibility distributions A and B for arbitrary weighting function $g(\gamma)$:

$$L_g(A,B) = \int_0^1 L(X_\gamma, Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma \approx 0.6772$$

$$\rho_g(A,B) = \int_0^1 \rho(X_\gamma, Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = -0.5$$

$$\eta_g(A,B) = \int_0^1 \eta(X_\gamma, Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = 0.5$$

C. Pyramidal joint possibility distribution

Let the joint possibility distribution be a pyramid, whose vertices are $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ on the xy-plane, see Fig.3. The marginal possibility distributions are

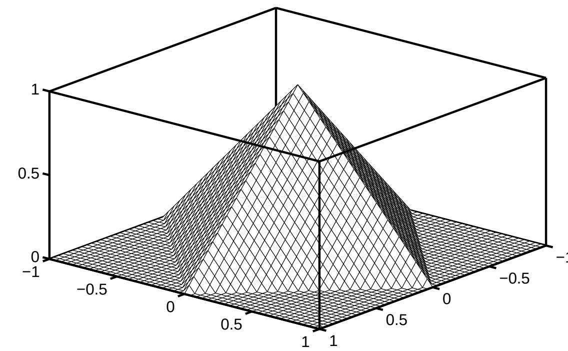

Fig. 3. Pyramid shaped joint possibility distribution. Because of symmetry the correlation coefficient and the correlation ratio are both zeros, but A and B are not independent.

$$A(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0 \ , \\ 1-x & \text{if } 0 < x \le 1 \ , \\ 0 & \text{otherwise}\,. \end{cases}$$

$$B(y) = \begin{cases} y+1 & \text{if } -1 \le y \le 0 \ , \\ 1-y & \text{if } 0 < y \le 1 \ , \\ 0 & \text{otherwise}\,. \end{cases}$$

Then the γ-level set is a square with vertices $(1-\gamma, 0)$, $(0, 1-\gamma)$, $(-(1-\gamma), 0)$ and $(0, -(1-\gamma))$.

Because of symmetry the correlation coefficients and the correlation ratio of X_γ and Y_γ are both zero, but X_γ and Y_γ are not independent:

$$f(x,y) \ne f_1(x)\cdot f_2(y)\,.$$

The area of the γ-level set:

$$T_\gamma = 2(1-\gamma)^2\,.$$

The marginal density function (we have the same expression for $f_2(y)$ with y):

$$f_1(x) = \begin{cases} \dfrac{1}{T_\gamma}\cdot 2(x+1-\gamma) & \text{if } -1+\gamma \le x \le 0 \ , \\ \dfrac{1}{T_\gamma}\cdot 2(1-\gamma-x) & \text{if } 0 \le x \le 1-\gamma \ , \\ 0 & \text{otherwise}\,. \end{cases}$$

We have to compute the following:

$$\int_{-(1-\gamma)}^{1-\gamma} f_1(x)\ln f_1(x)\,\mathrm{d}x =$$
$$= \int_{-(1-\gamma)}^{0} \frac{1}{T_\gamma}\cdot 2(x+1-\gamma)\ln\frac{1}{T_\gamma}\cdot 2(x+1-\gamma)\,\mathrm{d}x$$
$$+ \int_{0}^{1-\gamma} \frac{1}{T_\gamma}\cdot 2(1-\gamma-x)\ln\frac{1}{T_\gamma}\cdot 2(1-\gamma-x)\,\mathrm{d}x$$
$$= 2\cdot\left(\frac{1}{2}\ln\frac{1}{1-\gamma}-\frac{1}{4}\right) = \ln\frac{1}{1-\gamma}-\frac{1}{2}\,.$$

The mutual information:

$$I(X_\gamma,Y_\gamma) = \ln\frac{1}{T_\gamma} - 2\cdot\left(\ln\frac{1}{1-\gamma}-\frac{1}{2}\right)$$
$$= \ln\frac{1}{2(1-\gamma)^2} - 2\ln\frac{1}{1-\gamma}+1 = 1-\ln 2\,.$$

From this we get the informational coefficient of correlation which is the same as in the previous example:

$$L(X_\gamma,Y_\gamma) = \sqrt{1-e^{-2I(X_\gamma,Y_\gamma)}} \approx 0.6772\,.$$

In this case the correlation measures are not depend on the level γ, so the g-weighted measures of correlation between the marginal possibility distributions A and B for arbitrary weighting function $g(\gamma)$:

$$L_g(A,B) = \int_0^1 L(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma \approx 0.6772$$
$$\rho_g(A,B) = \int_0^1 \rho(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = 0$$
$$\eta_g(A,B) = \int_0^1 \eta(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = 0$$

D. *Conical joint possibility distribution*

Let the joint possibility distribution be a cone, whose axis is the z-axis, and the base is a circle with radius 1, see Fig.4. The marginal possibility distributions are:

$$A(x) = \begin{cases} x+1 & \text{if } -1\le x\le 0\ , \\ 1-x & \text{if } 0< x\le 1\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

$$B(y) = \begin{cases} y+1 & \text{if } -1\le y\le 0\ , \\ 1-y & \text{if } 0< y\le 1\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

Then the γ-level set is a circle with centre $(0,0)$ and radius $1-\gamma$. All of the γ-level sets are symmetrical to the x and the y axis, so in this case the correlation coefficient and the correlation ratio are both zero, although X_γ and Y_γ are not independent, because

$$f(x,y) \ne f_1(x)\cdot f_2(y)\,.$$

The area of the γ-level:

$$T_\gamma = \pi(1-\gamma)^2\,.$$

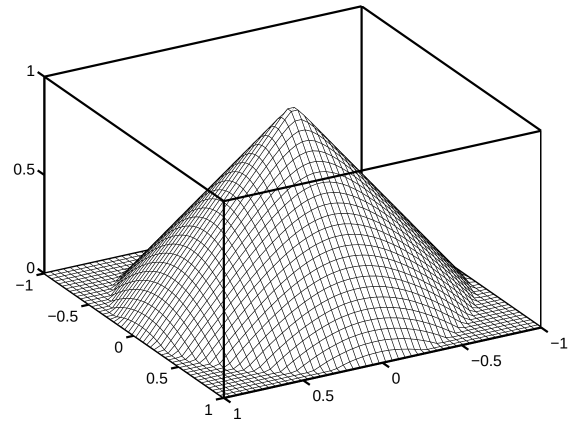

Fig. 4. Conical joint possibility distribution. Because of symmetry the correlation coefficient and the correlation ratio are both zeros, but A and B are not independent.

The marginal density function (we have the same expression for $f_2(y)$ with y):

$$f_1(x) = \begin{cases} \dfrac{1}{T_\gamma}\cdot 2\sqrt{(1-\gamma)^2-x^2} & \text{if } x\le|1-\gamma|\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

The integrals in this case are quite difficult, so we used numerical methods. The mutual information is approximately:

$$I(X_\gamma,Y_\gamma) = \ln\frac{1}{T_\gamma} - 2\int_{-(1-\gamma)}^{1-\gamma} f_1(x)\ln f_1(x)\,\mathrm{d}x \approx 0.1447\,.$$

With this approximation the informational coefficient of correlation is:

$$L(X_\gamma,Y_\gamma) = \sqrt{1-e^{-2I(X_\gamma,Y_\gamma)}} \approx 0.5013\,.$$

In this case the correlation measures are not depend on the level γ, so the g-weighted measures of correlation between the marginal possibility distributions A and B for arbitrary weighting function $g(\gamma)$:

$$L_g(A,B) = \int_0^1 L(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma \approx 0.5013$$
$$\rho_g(A,B) = \int_0^1 \rho(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = 0$$
$$\eta_g(A,B) = \int_0^1 \eta(X_\gamma,Y_\gamma)\cdot g(\gamma)\,\mathrm{d}\gamma = 0$$

E. *Larsen t-norm*

Let the joint possibility distribution defined by the product t-norm (Larsen t-norm, [22]), see Fig.5 (for more general case, when the joint possibility distribution is defined by the product of triangular fuzzy numbers see [12]):

$$C(x,y) = \begin{cases} xy & \text{if } 0\le x,y\le 1\ , \\ 0 & \text{otherwise}\,. \end{cases}$$

The marginal possibility distributions are

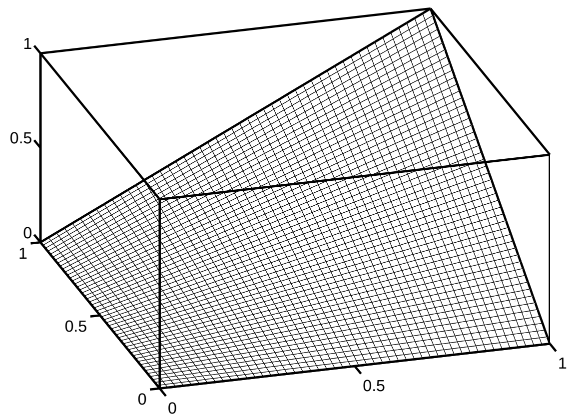

Fig. 5. Joint possibility distribution defined by the product t-norm (Larsen t-norm). In this case the correlation measures are depend on the γ-level sets. The possibilistic correlation can be determined by an appropriate weighting function.

$$A(x) = \begin{cases} x & \text{if } 0 \le x \le 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$$B(y) = \begin{cases} y & \text{if } 0 \le y \le 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then a γ-level set of C is,

$$[C]^\gamma = \big\{(x, y) \in \mathbb{R}^2 |\ 0 \le x, y \le 1,\ xy \ge \gamma\big\}.$$

The area of the γ-level:

$$T_\gamma = 1 - \gamma + \gamma \ln \gamma\,.$$

The marginal density function (we have the same expression for $f_2(y)$ with y):

$$f_1(x) = \begin{cases} \dfrac{1}{T_\gamma} \cdot \dfrac{x - \gamma}{x} & \text{if } \gamma \le x \le 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In this case the correlation coefficient, the correlation ratio, the mutual information and the informational coefficient of correlation depend on the level γ. The integrals are quite difficult (see [23]), so we used numerical integration methods. The results are shown on Fig.6. The limits in zero and in 1 are the following (see [23], [24])

$$\lim_{\gamma \to 0} \rho = 0\,, \qquad \lim_{\gamma \to 1} \rho = -1/2\,,$$
$$\lim_{\gamma \to 0} \eta = 0\,, \qquad \lim_{\gamma \to 1} \eta = 1/2\,,$$
$$\lim_{\gamma \to 0} L = 0\,, \qquad \lim_{\gamma \to 1} L \approx 0.6772\,.$$

In this case the measures of correlation depend on γ, so the weighted possibilistic correlation can be determined by choosing a weighting function $g(\gamma)$. If the weighting function is $g(\gamma) \equiv 1$ then:

$$L_g(A, B) = \int_0^1 L(X_\gamma, Y_\gamma) \cdot 1\, \mathrm{d}\gamma \approx 0.5786$$
$$\rho_g(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) \cdot 1\, \mathrm{d}\gamma \approx 0.3888$$
$$\eta_g(A, B) = \int_0^1 \eta(X_\gamma, Y_\gamma) \cdot 1\, \mathrm{d}\gamma \approx 0.4010$$

If the weighting function is $g(\gamma) = 2\gamma$ then:

$$L_g(A, B) = \int_0^1 L(X_\gamma, Y_\gamma) \cdot 2\gamma\, \mathrm{d}\gamma \approx 0.6282$$
$$\rho_g(A, B) = \int_0^1 \rho(X_\gamma, Y_\gamma) \cdot 2\gamma\, \mathrm{d}\gamma \approx 0.4441$$
$$\eta_g(A, B) = \int_0^1 \eta(X_\gamma, Y_\gamma) \cdot 2\gamma\, \mathrm{d}\gamma \approx 0.4490$$

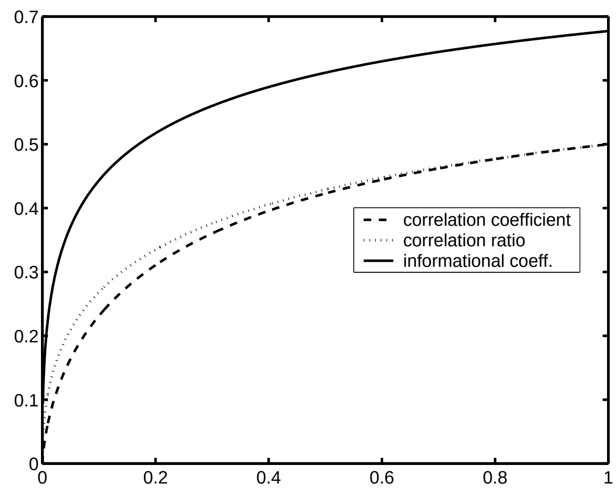

Fig. 6. The absolute value of the correlation coefficient, the correlation ratio and the informational coefficient of correlation in function of γ, when the joint possibility distribution is defined by Larsen t-norm.

VI. Conclusion

We have introduced the notion of weighted possibilistic informational coefficient of correlation between marginal distributions of a joint possibility distribution. This coefficient is equal to zero if and only if the marginal possibility distributions are independent (which does not hold for possibilistic correlation coefficient and ratio).

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