Periodic and antiperiodic eigenvalues for half-linear version of Hill's equation

Gabriella Bognár
University of Miskolc
Department of Analysis
Miskolc-Egyetemváros, Hungary 3515
Email: matvbg@uni-miskolc.hu

***Abstract*—The nonlinear eigenvalue problem of the differential equation $\left(|x'|^{p-2}\,x'\right)' + (\lambda + c(t))\,|x|^{p-2}\,x = 0,\ p > 1$, with respect to the periodic boundary conditions: $x(0) = x(T),\ \ x'(0) = x'(T)$, or to the antiperiodic boundary conditions: $x(0) = -x(T),\ \ x'(0) = -x'(T)$ are considered. Various results on the set of eigenvalues concerning both problems are presented. Some estimates are given for the periodic and antiperiodic eigenvalues.**

Key–Words: **Hill's equation, periodic solution, eigenvalues,**

I. Introduction

A Hill's equation is a differential equation of the type

$$x'' + q(t)\ x = 0,$$

where $q(t)$ is an integrable, real function of period p. This type of equation was first investigated in connection with the theory of lunar motion by G. W. Hill [9]. It is also well-known in the quantum theory of metals and semi-conductiors (see e.g., [4], [17] and [18]) or in optics when ultrashort optical pulses are examined (see e.g., [12], [15] and [16]). The value of the period of the solution plays an important role in the discussion of periodic solutions. A specific question is the case of solutions of period p and $2p$ (see [5], [7], [13]).

We consider the half-linear version of Hill's differential equation

$$x''\left|x'\right|^{p-2} + q(t)\ x\,|x|^{p-2} = 0, \quad p > 1. \tag{1}$$

It is called half-linear differential equation by I. Bihari [2]. Its solution set preserves the half of the properties of the linear differential equation since it is homogeneous, but not additive. In [8], Á. Elbert established the existence and uniqueness of solutions to the initial value problem for differential equation equation of type (1).

The aim of this paper is to examine the periodic solutions of equations (1) with periodic or antiperiodic boundary conditions

$$x(0) = x(T) \ \text{ and } \ x'(0) = x'(T), \tag{2}$$

or

$$x(0) = -x(T) \ \text{ and } \ x'(0) = -x'(T), \tag{3}$$

respectively, when $q(t) = \lambda + c(t)$, $\lambda \in R$, and the potential c, $t \in (0,T)$ is periodic. The value λ is called an eigenvalue and $x \neq 0$ an eigenfunction if the pair (λ, x) satisfies (1)-(2) or (1)-(3). We investigate the asymptotic behavior of large eigenvalues.

II. Preliminaries

In this section we recall some known results and techniques.

A. Generalized sine function

For the special case, $q(t) \equiv 1$, the solution of equation

$$x''\left|x'\right|^{p-2} + x\,|x|^{p-2} = 0 \tag{4}$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$, called the generalized sine function

$$x = S_p(t), \quad t \in (-\infty, +\infty) \tag{5}$$

was introduced by Á. Elbert in [8]. For $t \in [0, \widehat{\pi}/2]$, where

$$\widehat{\pi}/2 = \frac{\pi}{p} / \sin\frac{\pi}{p},$$

function S_p satisfies

$$t = \int_0^{S_p} \frac{dx}{\sqrt[p]{1 - x^p}}. \tag{6}$$

Formula (6) defines uniquely function S_p on $[0, \widehat{\pi}/2]$ with $S_p\left(\widehat{\pi}/2\right) = 1$.

We extend S_p to all $\mathbf{R}$ (and still denote this extension by S_p) as a $2\widehat{\pi}$ periodic function:

$$\begin{aligned} S_p(t) &= S_p(\widehat{\pi} - t) \quad \text{for } t \in [\widehat{\pi}/2, \widehat{\pi}]\,, \\ S_p(t) &= -S_p(-t) \quad \text{for } t \in [-\widehat{\pi}, 0]\,, \\ S_p(t) &= S_p(t + 2\widehat{\pi}) \quad \text{for } t \in \mathbf{R}. \end{aligned} \tag{7}$$

Therefore function S_p has the following properties:
(i) $S_p(t + \hat{\pi}) = -S_p(t)$ for all t and $S_p(t)$ is an odd function having zeros at $t = j\hat{\pi},\ j = \mathbf{Z}$,
(ii) $S_p'(t)$ has zeros only at $t = \frac{1}{2}\hat{\pi} + j\hat{\pi},\ j = \mathbf{Z}$.
(iii) From (4) by integration we have the generalized Pythagorean relation

$$|S_p(t)|^p + \left|S_p'(t)\right|^p = 1 \quad \text{for all } \ t \in \mathbf{R}. \tag{8}$$

For $p = 2$ we have that

$$\begin{aligned} S_2(t) &= \sin t, \\ \hat{\pi} &= \pi \end{aligned}$$

and equation (8) is reduced to the usual Pythagorean relation

$$\sin^2 t + \cos^2 t = 1.$$

B. Generalized Prüfer transformation

It is convenient to introduce the generalized Prüfer transformation for the examination of the solutions of the quasilinear differential equation (1) using the above defined generalized trigonometric function.

For $x(t, \lambda)$ of (1) the generalized polar functions $\varphi(t, \lambda)$ and $\rho(t, \lambda)$ are defined by

$$x(t,\lambda) = \rho(t,\lambda)\ S_p\left(\varphi(t,\lambda)\right),$$
$$x'(t,\lambda) = \rho(t,\lambda)\ S_p'\left(\varphi(t,\lambda)\right),$$

where

$$\rho(t,\lambda) = \left[|x(t,\lambda)|^p + \left|x'(t,\lambda)\right|^p\right]^{1/p}$$

moreover $\varphi(t,\lambda)$ and $\rho(t,\lambda)$ are continuously differentiable functions of t. Then the pair

$$(\varphi;\rho) = (\varphi(t,\lambda);\rho(t,\lambda))$$

is a solution of the system of differential equations

$$\varphi' = \left|S_p'(\varphi)\right|^p + \frac{q(t)}{p-1}\left|S_p(\varphi)\right|^p,$$
$$\rho' = \rho\left(1-\frac{q(t)}{p-1}\right)S_p'(\varphi)\left|S_p(\varphi)\right|^{p-2}S_p(\varphi).$$

III. Periodic and antiperiodic eigenvalue problems

We consider differential equation (1) for $q(t) = \lambda + c(t)$:

$$\left(\left|x'\right|^{p-2}x'\right)' + (\lambda + c(t))\left|x\right|^{p-2}x = 0 \qquad (9)$$

in $(0,T)$, where $T>0$, $p>1$ is real number, $c(t)$ is a positive, continuous and periodic function on $(0,T)$. The boundary conditions are

$$x(0) = x(T),\ \ x'(0) = x'(T), \qquad \text{(P)}$$

called periodic boundaryconditions, or

$$x(0) = -x(T),\ \ x'(0) = -x'(T) \qquad \text{(AP)}$$

called antiperiodic conditions.

Let $x = x(t)$ be a solution of (9) with (P). We extend x as a periodic function on $\mathbf{R}$ such as

$$x(t+T) = x(t) \text{ for any } t\in\mathbf{R},$$

then x is a T-periodic solution of (9) on the whole of $\mathbf{R}$.

Let $\tilde{x} = \tilde{x}(t),\ t\in[0,T]$ be a solution of (9) with (AP) and extend x as follows:

$$\tilde{x}(t) = -\tilde{x}(t-T) \text{ for } t\in(T,2T),$$

then

$$\tilde{x}(t) = \tilde{x}(t+2T) \text{ for any } t\in\mathbf{R}.$$

Therefore $\tilde{x}$ is a $2T$-periodic solution of (9) on the whole $\mathbf{R}$.

For the functional settings we define $W_T^{1,p}$ as a function space of all continuous functions $y = y(t)$, $t\in[0,\ T]$, such that

$$\|y\| = \left(\int_0^T\left[\left|y'\right|^p + c(t)\left|y\right|^p\right]dt\right)^{1/p} < \infty$$

and y satisfies (P).

Let us define $\tilde{W}_T^{1,p}$ as a function space of all continuous functions $y = y(t)$, $t\in[0,\ T]$ such that $\|y\|<\infty$ and y satisfies (AP).

Both $W_T^{1,p}$ and $\tilde{W}_T^{1,p}$ are Banach spaces (Sobolev spaces of T-periodic and T-antiperiodic functions) and $\|.\|$ defines a norm in both spaces.

We can summarize the following properties:

$$\forall x \in W_T^{1,p} \Rightarrow |x|\in W_T^{1,p},$$
$$\forall x \in \tilde{W}_T^{1,p} \Rightarrow |x|\notin \tilde{W}_T^{1,p},$$
$$\forall x \in \tilde{W}_T^{1,p} \Rightarrow |x|\in W_T^{1,p},$$

moreover, we have that

$$W_T^{1,p}\cap\tilde{W}_T^{1,p} = \left\{x : x(0) = x(T) = x'(0) = x'(T) = 0\right\}.$$

The formulation of the eigenvalue problem of (9)-(P) in a weak sense is the following:

Definition 1 *Function $x\in W_T^{1,p}$ is called the weak solution of (9)-(P) if for all $y\in W_T^{1,p}$*

$$\int_0^T\left|x'\right|^{p-2}x'y'dt - (\lambda + c(t))\int_0^T|x|^{p-2}xydt = 0$$

is satisfied.

Analogously, we have

Definition 2 *The weak solution of (9)-(AP) is a function $\tilde{x}\in\tilde{W}_T^{1,p}$ if*

$$\int_0^T\left|\tilde{x}'\right|^{p-2}\tilde{x}'y'dt - (\lambda + c(t))\int_0^T|\tilde{x}|^{p-2}\tilde{x}ydt = 0$$

holds for all $y\in\tilde{W}_T^{1,p}$.

The regularity of the weak solution can be considered by „standard" regularity argument given by M. Otani [14]. If x is a weak solution of (9)-(P) then $x\in C^1[0,T]$. We have the same property for $\tilde{x}$. Moreover, if x is a weak solution of (9)-(P) then $x\in C^2(0,T)$ with exception of the points t where $x'(t) = 0$ for $p>2$. (The same holds for $\tilde{x}$.)

We note that the initial value problem of (9) under initial conditions

$$x(t_0) = x_0,\ x'(t_0) = x_1$$

admits unique solution $x\in C^1(\mathbf{R})$, and $x\in C^2(\mathbf{R})$ with exception of the points where $x' = 0$ for $p>2$ (see [6]).

For the variational characterization of the eigenvalues we set

$$\|y\|_p = \left(\int_0^T|y|^p\,dt\right)^{1/p}$$

for $y\in W_T^{1,p}\cup\tilde{W}_T^{1,p}$, and we use the notation

$$S := \left\{y\in W_T^{1,p} : \|y\|_p = 1\right\}.$$

For a closed and symmetric set $A\subset S$ we define the Krasnoselski genus of A as follows:

$$\gamma(A) := \inf\{m\in N :$$
$$\exists\ \text{ continuous and odd mapping } A \text{ into } \mathbf{R}^m\setminus\{0\}\},$$

and

$$\gamma(A) := \infty \quad \text{if such } \ m \text{ does not exist.}$$

Let us define

$$F_k := \{A\subset S :\ A = -A,\ \gamma(A) = k\},\ k\in N.$$

We denote by $\tilde{S}, \tilde{A}, \tilde{F}_k$ the same sets if $W_T^{1,p}$ is replaced by $\tilde{W}_T^{1,p}$. The eigenvalues of (9)-(P) (or (9)-(AP)) are those values of $\lambda \in R$ for which there exists non-zero solution of (9)-(P) (or (9)-(AP)).

Definition 3. *Let us denote by λ_k and $\tilde{\lambda}_k$ the eigenvalues of (9)-(P) and (9)-(AP). Then we have*

$$\lambda_k := \min_{A \in F_{k+1}} \max_{x \in A} \|x\|^p \ \text{ for } k \in \{0\} \cup \mathbf{N},$$

and

$$\tilde{\lambda}_k := \min_{\tilde{A} \in \tilde{F}_{k+1}} \max_{x \in \tilde{A}} \|x\|^p \ \text{ for } k \in \mathbf{N}.$$

We consider the eigenvalues of (9) with respect to the periodic (P) or antiperiodic (AP) boundary conditions. When no potentials are present, $c(t) \equiv 0$, the periodic and antiperiodic eigenvalues of (9) are known because (9) is integrable. Thus the eigenvalue problem

$$\left(\left|x'\right|^{p-2} x'\right)' + \lambda \left|x\right|^{p-2} x = 0,$$
$$x(0) = x(\hat{\pi}) = 0$$

has a solution

$$x = C \ S_p\left(\sqrt[p]{\lambda} t\right), \text{ for } C \in \mathbf{R},$$

which is a periodic solution. In order to obtain nonvanishing solutions it is necessary that

$$\sqrt[p]{\lambda} = n, \ \ n = 1, 2, 3, ...,$$

and the eigenfunctions are given by

$$x_n = C_n \ S\left(n \ t\right).$$

If $p = 2$, $c(t)$ is 2π periodic and $c \in L^1(0, 2\pi)$ (this is the case $T = 2\pi$) then the classical results are known (see e.g., [13]). However, when some potentials are present in (9), $c(t) \neq 0$, the periodic and antiperiodic eigenvalues are studied by M. Zhang [19].

It is known that there exist two sequences $\left\{\lambda_k : \ (k \in \mathbf{Z}^+)\right\}$ and $\left\{\tilde{\lambda}_k : \ (k \in \mathbf{N})\right\}$ of the reals such that

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 < \dots,$$
$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 < \tilde{\lambda}_3 \leq \tilde{\lambda}_4 < \tilde{\lambda}_5 \leq \tilde{\lambda}_6 < \dots$$

and both sequences $\left\{\lambda_k : \ (k \in \mathbf{Z}^+)\right\}$ and $\left\{\tilde{\lambda}_k : \ (k \in \mathbf{N})\right\}$ tend to $+\infty$ as $k \to +\infty$.

It is also known that the number of nodes of x_k (or of $\tilde{x}_k$) in $[0, T)$ is finite. Additionally we can give the number of nodes in the two cases. Let x_k be the eigenfunction associated with λ_k ($k = 0, 1, 2, \ldots$). Then the number of nodes of x_k in $[0, T)$ for $k = 2n$ is equal to $2n$ ($n = 0, 1, 2, \ldots$) and for $k = 2n-1$ is equal to $2n$ ($n = 0, 1, 2, \ldots$).

The smallest eigenvalue λ_0 is simple and isolated. (the proof is similar as in [1]).

Let $\tilde{x}_k$ be the eigenfunction associated with $\tilde{\lambda}_k$ ($k = 1, 2, \ldots$). Then the number of nodes of $\tilde{x}_k$ in $[0, T)$ is $2n-1$ for $k = 2n-1$ ($n = 1, 2, \ldots$) and $2n-1$ also for $k = 2n$ ($n = 1, 2, \ldots$). Here the smallest eigenvalue $\tilde{\lambda}_1$ is not simple in general. It is enough to take the linear case ($p = 2$) with $c(t) \equiv const.$ when $\tilde{\lambda}_1 = \tilde{\lambda}_2$.

IV. Asymptotic results

Henceforth we consider differential equation (9) for sufficiently large value of λ such that

$$\lambda + c(t) > 0. \tag{10}$$

Without loss of generality we assume that $c(t)$ is integrable and

$$\int_0^{\hat{\pi}} c(t) \ dt = 0.$$

Let c in (9) be a periodic function of t with period $\hat{\pi}$ and let $c(t)$ be satisfy

$$\left[\lambda + c(t)\right]^{1+1/p} > \frac{1}{p}\left|c'(t)\right|, \quad \text{for all} \quad t. \tag{11}$$

To emphasize the dependence solution of (9) on λ we shall write $x(t, \lambda)$.

First we construct a solution y of (9) such that

$$x(t, \lambda) = A(t) \ S_p\left(\varphi(t)\right) \tag{12}$$

and

$$x'(t, \lambda) = \sqrt[p]{\lambda + c(t)} \ A(t, \lambda) \ S_p'\left(\varphi(t, \lambda)\right), \tag{13}$$

where $\varphi(t)$ and $A(t)$ are continuously differentiable on $[0, \infty)$ and determined by the differential equations

$$\varphi'(t, \lambda) = \sqrt[p]{\lambda + c(t)} + \frac{1}{p} \ \frac{c'(t)}{\lambda + c(t)} \ G(\varphi). \tag{14}$$

$$\frac{A'(t, \lambda)}{A(t, \lambda)} = -\frac{1}{p} \ \frac{c'(t)}{\lambda + c(t)} \ \left|S_p\left(\varphi(t, \lambda)\right)\right|^p, \tag{15}$$

with notation

$$G(\varphi) = S_p\left(\varphi\right) \ \left|S_p'\left(\varphi\right)\right|^{p-2} \ S_p'\left(\varphi\right).$$

The conditions on $x(0)$ and $x'(0)$ determine the values of $\varphi(0)$ and $A(0)$.

Inequality (11) guarantees that $\varphi(t, \lambda)$ is monotonically increasing function of t. From (7) it follows that if $\varphi(t)$ and $A(t)$ provide a solution $x(t, \lambda)$ then $\varphi(t) + \hat{\pi}$ and $A(t)$ also provide as a solution $-x(t, \lambda)$. All the solutions can be obtained on the range of values $\varphi(0)$ where the range is of length $\hat{\pi}$. We get from (15) that function

$$A(t, \lambda) = A(0, \lambda) \exp \alpha(t),$$

with

$$\alpha(t) = \left(-\frac{1}{p}\int_0^t \frac{c'(\tau)}{\lambda + c(\tau)} \ \left|S_p\left(\varphi(\tau, \lambda)\right)\right|^p d\tau\right)$$

is monotone, non-increasing and tends to a limit $A(\infty, \lambda)$ as $t \to \infty$. If

$$\int_0^\infty \frac{c'(\tau)}{\lambda + c(\tau)} \ \left|S_p\left(\varphi(\tau, \lambda)\right)\right|^p d\tau = \infty$$

then $A(\infty, \lambda) = 0$ and

$$\lim_{t \to \infty} x(t, \lambda) = 0.$$

If

$$\int_0^\infty \frac{c'(\tau)}{\lambda + c(\tau)} \ \left|S_p\left(\varphi(\tau, \lambda)\right)\right|^p d\tau < \infty$$

then $A(\infty, \lambda) > 0$ and solution $x(t, \lambda)$ oscillates, where its amplitude tends to a positive value.

If $x(t,\lambda)$ is periodic with period $\hat{\pi}$, then we get from (12) and (13) with conditions

$$\begin{aligned} x(\hat{\pi},\lambda) &= x(0,\lambda), \\ x'(\hat{\pi},\lambda) &= x'(0,\lambda) \end{aligned} \tag{16}$$

that

$$\begin{aligned} A(\hat{\pi},\lambda) &= A(0), \\ \varphi(\hat{\pi},\lambda)-\varphi(0,\lambda) &= 2k\ \hat{\pi}, \end{aligned} \tag{17}$$

where k is a positive integer.

If $x(t,\widetilde{\lambda})$ is antiperiodic, then for the solution we have conditions

$$\begin{aligned} x(\hat{\pi},\widetilde{\lambda}) &= -x(0,\widetilde{\lambda}), \\ x'(\hat{\pi},\widetilde{\lambda}) &= -x'(0,\widetilde{\lambda}), \end{aligned} \tag{18}$$

hence

$$\begin{aligned} A(\hat{\pi},\widetilde{\lambda}) &= A(0,\widetilde{\lambda}), \\ \varphi(\hat{\pi},\widetilde{\lambda})-\varphi(0,\widetilde{\lambda}) &= (2k-1)\ \hat{\pi}, \end{aligned} \tag{19}$$

where k is a positive integer.

Problem (9) with

$$x(\hat{\pi},\lambda)=x(0,\lambda),\quad x'(\hat{\pi},\lambda)=x'(0,\lambda)$$

has countable infinity of values

$$\lambda_0\le\lambda_1\le\lambda_2\le ...\le\lambda_k\le ...$$

accumulating at ∞, similarly, for problem (9) with

$$x(\hat{\pi},\widetilde{\lambda})=-x(0,\widetilde{\lambda}),\quad x'(\hat{\pi},\widetilde{\lambda})=-x'(0,\widetilde{\lambda})$$

has countable infinity of values

$$\widetilde{\lambda}_1\le\widetilde{\lambda}_2\le ...\le\widetilde{\lambda}_k\le ...$$

accumulating at ∞ and for each nonnegative integer

$$\begin{aligned} \lambda &= \lambda_k \ \text{ with } \ \varphi(\hat{\pi},\lambda_k)-\varphi(0,\lambda_k)=2k\ \hat{\pi}, \\ \widetilde{\lambda} &= \widetilde{\lambda}_k \ \text{ with } \ \varphi(\hat{\pi},\widetilde{\lambda}_k)-\varphi(0,\widetilde{\lambda}_k)=(2k-1)\ \hat{\pi} \end{aligned}$$

for the proof see [19].

Now we gain more information regarding the distribution of the parameter λ. We give estimates for large eigenvalues:

Theorem 4 *Let $c(t)$, $c'(t)$ and $c''(t)$ be bounded periodic functions with period $\hat{\pi}$. Then for λ_k and $\widetilde{\lambda}_k$ concerning the solutions of (9)-(16) and (9)-(18), respectively,*

$$\begin{aligned} \sqrt[p]{\lambda_{2k}}-2k &= O\left(\tfrac{1}{k^\nu}\right), \\ \sqrt[p]{\lambda_{2k-1}}-2k &= O\left(\tfrac{1}{k^\nu}\right), \\ \sqrt[p]{\widetilde{\lambda}_{2k}}-(2k-1) &= O\left(\tfrac{1}{k^\nu}\right), \\ \sqrt[p]{\widetilde{\lambda}_{2k-1}}-(2k-1) &= O\left(\tfrac{1}{k^\nu}\right) \end{aligned} \tag{20}$$

hold for large values of k with

$$\nu=\begin{cases} 2p-1, & \text{if} \quad 1<p<2, \\ p+1, & \text{if} \quad p\ge 2. \end{cases}$$

Proof: For $t_0=0$ we get a Volterra–type integral equation for φ

$$\begin{aligned} \varphi(t,\lambda) &= \varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau \\ &\quad +\frac{1}{p}\int_0^t\frac{c'(\tau)}{\lambda+c(\tau)}\,G(\varphi)\,d\tau. \end{aligned} \tag{21}$$

Since $c'(t)$ and $c(t)$ are bounded, and if λ is large enough, then

$$\frac{1}{p}\left|\int_0^t\frac{c'(\tau)}{\lambda+c(\tau)}\,G(\varphi)\,d\tau\right|<\frac{K}{\lambda}$$

$K=const.$, and

$$\varphi(t,\lambda)=\varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau+O\left(\frac{1}{\lambda}\right),$$

As for sufficiently large λ

$$\begin{aligned} S_p(\varphi) &= S_p\left(\varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau\right)+O\left(\tfrac{1}{\lambda}\right), \\ S_p'(\varphi) &= S_p'\left(\varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau\right)+O\left(\tfrac{1}{\lambda}\right), \end{aligned} \tag{22}$$

so that

$$G(\varphi)=G\left(\varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau\right)+O\left(\frac{1}{\lambda}\right). \tag{23}$$

By an iteration we find that

$$\begin{aligned} \varphi(t,\lambda) &= \varphi(0,\lambda)+\int_0^t\sqrt[p]{\lambda+c(\tau)}\,d\tau \\ &+\frac{1}{p}\int_0^t\frac{c'(\tau)}{\lambda+c(\tau)}\,G\left(\varphi(0,\lambda)+\int_0^\tau\sqrt[p]{\lambda+c(s)}\,ds\right)\,d\tau \\ &+O\left(\frac{1}{\lambda^2}\right). \end{aligned}$$

For $\hat{\pi}$ periodic solution we get

$$x(0)=x(\hat{\pi}),\quad x'(0)=x'(\hat{\pi})$$

$$A(\hat{\pi},\lambda)=A(0),\quad \varphi(\hat{\pi},\lambda)-\varphi(0,\lambda)=2k\,\hat{\pi},$$

$$\begin{aligned} 2k\,\hat{\pi} &= \int_0^{\hat{\pi}}\sqrt[p]{\lambda_k+c(\tau)}\,d\tau \\ &\quad +\frac{1}{p}\int_0^{\hat{\pi}}\frac{c'(\tau)}{\lambda_k+c(\tau)}\,G\left(\varphi(\tau,\lambda_k)\right)\,d\tau, \end{aligned} \tag{24}$$

$$0=-\frac{1}{p}\int_0^{\hat{\pi}}\frac{c'(\tau)}{\lambda_k+c(\tau)}\,\left|S_p\left(\varphi(\tau,\lambda_k)\right)\right|^p d\tau. \tag{25}$$

The values of $\varphi(0,\lambda_k)$ and λ_k are unknown. As $\varphi(t,\lambda_k)$ is determined from (21) then $\varphi(0,\lambda_k)$ and λ_k can be determined from (24) and (25) for every k**.** Applying (22) and (23) we obtain estimates on λ_k and $\tilde{\lambda}_k$.

Theorem 5. *Let $c(t)$ be periodic with period $\hat{\pi}$ and let M be a uniform bound for $|c|$, $|c'|$, $|c''|$, and $|c'''|$. Then the eigenvalues λ and $\tilde{\lambda}$ belonging to the problem (9)-(P) and (9)-(AP) when $p\ne 3$ satisfy the inequalities*

$$\begin{aligned} \sqrt[p]{\lambda_{2k}}>2k, &\qquad \sqrt[p]{\lambda_{2k-1}}>2k, \\ \sqrt[p]{\tilde{\lambda}_{2k}}>(2k-1), &\qquad \sqrt[p]{\tilde{\lambda}_{2k-1}}>(2k-1) \end{aligned}$$

provided that they are greater than constant Λ defined by

$$\Lambda = \max\left(M + \left(1 + \frac{M}{p^2}\right)^p,\right.$$
$$\left. M + \left(\frac{2p}{p-1} \cdot \frac{C_1 + C_2 M + C_3 M^2}{M}\right)^{\frac{p}{3-p}}\right),$$

where $C_1 = C_1(p)$, $C_2 = C_2(p)$, $C_3 = C_3(p)$.

For the proof we refer [3].

Remark. The bound obtained for the Hill's equation (equation (9) with $p = 2$) by H. Hochstadt [10] is better than our bound. The reason is that in the linear case we are able to use trigonometric formulas but if $p \neq 2$ then these formulas do not exist for the generalized trigonometric functions.

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